On the Exact Obstruction to the Jamshidian Decomposition in Multifactor Affine Models

Aryan Ayyar
Fellow-in-Residence, MAHE Bengaluru
ayyar.aryan@manipal.edu

March 29, 2026

Abstract

The Jamshidian decomposition reduces a European coupon-bond option to a portfolio of zero-coupon bond options through a single deterministic set of component strikes. Its classical proof relies on every constituent bond price being monotone in the same scalar state variable, a property inherent to one-factor models but absent from genuinely multifactor affine term structure models. We provide three groups of results. First, an exact characterization: within the exponential-affine class $P_i(x) = A_i e^{-B_i^\top x}$, a deterministic-strike Jamshidian decomposition exists if and only if all loading vectors $\{B_i\}$ are positively collinear, i.e. lie on a single ray in $\mathbb{R}^d$ (Theorem 5.1). The proof rests on a Simultaneous Threshold Lemma that translates the pathwise identity into a comonotonicity condition, and a Cauchy–Schwarz perturbation argument that shows non-collinear loadings always violate comonotonicity near the exercise boundary. Second, a quantitative near-collinearity theory: we introduce the transverse dispersion $D_u := \max_{i,j} \|r_i/b_i - r_j/b_j\|$, where $B_i = b_i u + r_i$ is the orthogonal decomposition relative to a reference direction u, and prove that (i) the deterministic-strike approximation is a pathwise upper bound, with error confined to a strip of width $O(D_u\|z\|)$ around the projected one-factor exercise hyperplane; (ii) the projected strikes $\kappa_i^u := A_i e^{-b_i y^*}$ are the unique minimax-optimal deterministic strikes; and (iii) under a mild bounded-density condition, the pricing error satisfies $0 \leq V_u - V = O(D_u^2) = O(\delta^2)$ as $\delta = \max_{i,j} \angle(B_i, B_j) \downarrow 0$, a quadratic rate that is sharp by an explicit two-factor example. Third, structural impossibility extensions: unrestricted state-dependent strikes yield a trivially vacuous decomposition with no structural content; numéraire changes leave the exercise geometry invariant and cannot resolve the multifactor obstruction; and a scalar-factor rigidity theorem establishes that no change of state variable, linear or nonlinear, can restore comonotonicity when the loading vectors fail to be positively collinear.

Keywords: Jamshidian decomposition, affine term structure models, coupon bond options, positive collinearity, scalar-factor rigidity, transverse dispersion, minimax-optimal strikes, pathwise comonotonicity, angular spread.

MSC 2020: 91G30, 60G99, 91G20, 52A40.

1. Introduction

In his celebrated 1989 paper, Farshid Jamshidian observed that, under a one-factor interest rate model, every zero-coupon bond price at option expiry is a strictly monotone function of the same scalar state variable. This single observation reduces a European option on a coupon bond, formally a complex, path-dependent payoff to a finite sum of European options on individual zero-coupon bonds, each with a deterministic component strike that can be computed by solving a single scalar equation. The argument is pathwise: the identity holds state by state, not merely in expectation. The practical power of this decomposition is substantial. In the Vasicek (1977) and Hull–White (1990) models, zero-coupon bond options admit closed-form Black-like formulae, so Jamshidian's identity reduces coupon-bond option pricing to a finite sum of closed forms. The same device underlies the standard market approach to swaption pricing and the calibration of callable bond models. The decomposition is, however, well-known to fail in multifactor models. In a two-factor model, for instance, the state vector $x \in \mathbb{R}^2$ admits perturbations that push some bond prices above their threshold while pushing others below it, so no common exercise region exists. Practitioners respond either by numerical integration over the full state-space distribution, or by one-factor approximations that sacrifice pathwise exactness. The theoretical literature has noted this failure without characterizing it precisely: the standard treatment simply assumes a one-factor model and moves on.

We provide several results. *First*, a complete characterization of the exact decomposition (see Theorem 5.1) within the exponential-affine pricing family, a classical deterministic-strike Jamshidian decomposition exists *if and only if* all payment-date loading vectors $\{B_i\}$ lie on a single ray in $\mathbb{R}^d$. We also show that the proof rests on a Simultaneous Threshold Lemma (Lemma 4.1) that translates the pathwise identity into a comonotonicity condition, and a Cauchy–Schwarz perturbation argument that shows non-collinear loadings always violate comonotonicity near the exercise boundary. *Second*, we introduce a quantitative theory of near-collinear approximation (see Section 6). In this, we fix a reference direction u and decompose $B_i = b_i u + r_i$ orthogonally. The transverse dispersion

$$D_u := \max_{i,j} \left| \frac{r_i}{b_i} - \frac{r_j}{b_j} \right|$$

vanishes exactly in the collinear case and is $O(\delta)$ for small angular spread $\delta := \max_{i,j} \angle(B_i, B_j)$. We also prove:

(i) (Theorem 6.3) The deterministic-strike approximation is a pathwise upper bound, with error confined to a strip of width $O(D_u|z|)$ around the projected exercise hyperplane.

(ii) (Proposition 6.4) The *projected strikes* $\kappa_i^u := A_i e^{-b_i y^*}$—the natural one-factor construction along u, are the unique minimizers of the worst-case strip width over all deterministic strike vectors.

(iii) (Theorem 6.7) Under a mild bounded-density assumption, the pricing error satisfies $0 \leq V_u - V = O(D_u^2) = O(\delta^2)$, and this quadratic rate is sharp (Example 6.9).

Throughout, $B_i \neq 0$ and all component strikes are deterministic. Section 3 fixes notation. Section 4 proves the Simultaneous Threshold Lemma. Section 5 contains the exact characterization theorem. Section 6 develops the near-collinearity theory. Section 9 discusses open problems.

2. Literature Review

The present paper sits at the intersection of two bodies of work: the theory of affine term structure models and the analytical pricing of coupon-bond options. We survey both strands and then collect the mathematical results that underpin our proofs.

2.1. One-Factor Affine Short-Rate Models

The modern theory of bond pricing begins with *Vasicek* (1977), who modelled the instantaneous short rate as an Ornstein–Uhlenbeck process and derived the first closed-form affine bond-pricing formula under no-arbitrage Vasicek (1977). The key structural property of Vasicek's model is that, because the short rate is the unique state variable, every zero-coupon bond price is a strictly monotone function of the same scalar; this single-factor monotonicity is the essential premise on which Jamshidian's (1989) decomposition rests. *Cox, Ingersoll, and Ross* (1985) proposed a square-root diffusion for the short rate, ensuring non-negativity of interest rates and yielding bond prices that are again exponential-affine in the state variable Cox, Ingersoll and Ross (1985). Although the state space is $(0, \infty)$ rather than $\mathbb{R}$, the exponential-affine structure is preserved; our framework and proofs accommodate CIR-type models under the open-domain assumption of Remark 3.2.
Hull and White (1990) extended both the Vasicek and CIR models to be consistent with the observed initial term structure by introducing time-inhomogeneous parameters Hull and White (1990). In doing so they verified that Jamshidian's decomposition continues to apply in the Hull–White setting, since all bond prices remain strictly monotone in the single factor. Their analysis reinforces that the Jamshidian trick is essentially a one-factor phenomenon, which is the starting point of the investigation undertaken here.

2.2. Multifactor Affine Term Structure Models

A systematic multifactor generalisation was provided by *Duffie and Kan* (1996), who studied yield-factor models in which the vector of yields at selected maturities constitutes the state vector Duffie and Kan (1996). They showed that, within this class, zero-coupon bond prices are exponential-affine in the factors and that the loading coefficients A and B satisfy a system of Riccati ordinary differential equations. Assumption 3.1 of the present paper adopts this exponential-affine structure directly: $P_i(x) = A_i e^{-B_i^\top x}$, making the Duffie–Kan framework the natural ambient class for our results. *Dai and Singleton* (2000) undertook a comprehensive classification of affine term structure models (ATSMs), organising them into canonical families $\mathcal{A}_m(N)$ according to the number of state variables that enter the conditional variance Dai and Singleton (2000). Their analysis established that the loading vectors B_i carry the entire maturity dependence of the exponential-affine representation and that non-collinear loadings are the generic situation in models with two or more factors, a point directly relevant to Theorem 5.1 and Corollary 5.2 herein.

The rigorous mathematical foundations for affine processes were laid by *Duffie, Filipović, and Schachermayer* (2003), who provided a complete characterisation of regular affine Markov processes on $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, proving that the Laplace transform is exponential-affine and establishing when the Riccati ODEs yield globally defined solutions Duffie, Filipović and Schachermayer (2003). This foundational result justifies the global validity of $P_i(x) = A_i e^{-B_i^\top x}$ on the connected open state space E assumed in Assumption 2.1, and provides the analytic backdrop for both the exact obstruction theorem (Section 4) and the near-collinearity bounds (Section 5).

2.3. Bond Option Pricing and the Jamshidian Decomposition

The central reference for the present paper is *Jamshidian* (1989), who derived a closed-form solution for European options on zero-coupon bonds in the Vasicek model and, crucially, showed that an option on a coupon bond can be decomposed into a portfolio of options on constituent zero-coupon bonds, each with a deterministic component strike Jamshidian (1989). The argument exploits the strict monotonicity of every bond price in the single factor: at the unique short-rate threshold r^* where the coupon bond equals the option strike, each zero-coupon bond equals its component strike, and the payoff identity follows pathwise. Theorem 4.1 of the present paper identifies positive collinearity of the loading vectors $\{B_i\}$ as the precise algebraic condition that is both necessary and sufficient for this pathwise identity to hold, thereby sharpening Jamshidian's sufficient condition (which was formulated for a specific parametric model) into a model-free characterisation. *Geman, El Karoui, and Rochet* (1995) developed the general change-of-numéraire technique and showed how a judicious choice of numéraire simplifies option pricing under stochastic interest rates Geman, El Karoui and Rochet (1995). Their forward-measure framework underlies the pricing representation $V = P(0, T_0)\,\mathbb{E}^{T_0}[(C(X_{T_0}) - K)^+]$ used in Section 6.4, and is the natural measure-theoretic setting in which the pricing error bound of Theorem 6.7 is stated. *Wei* (1997) proposed a simplified approach to pricing European options on bond portfolios in a one-factor framework by matching the duration of the coupon bond to that of a proxy zero-coupon bond, thus reducing a multi-option problem to a single-option calculation Wei (1997). Wei's approach is complementary to Jamshidian's: where Jamshidian decomposes the option exactly into n constituent options, Wei approximates the portfolio by a single discount bond. Wei acknowledged that the procedure is an approximation (incurring errors from volatility-curve mismatch) but demonstrated that it performs well for standard parameter values in the Vasicek and CIR models. The approximation error in Wei's approach is structurally related to, yet distinct from, the transverse-dispersion error D_u studied in Section 6 of the present paper.

2.4. Coupon-Bond Options in Multifactor Affine Models

The literature most directly related to the present work is *Singleton and Umantsev* (2002), who derived a numerically accurate and computationally fast approximation for the prices of European options on coupon-bearing bonds in general affine term structure models Singleton and Umantsev (2002). Their approach linearises the exercise boundary in the factor space: they project the multi-dimensional exercise region onto a one-dimensional direction, i.e. the direction that carries the most variance and thereby reduce the option price to an expression involving the conditional characteristic function of the state vector, computable via standard Fourier transform inversion. This projection is structurally equivalent to the "projected strikes" $\kappa_i^u := A_i e^{-b_i y^*}$ of Section 6.3, and Singleton and Umantsev's empirical accuracy supports the $\mathrm{O}(\delta^2)$ error bound established rigorously in Theorem 6.7. However, Singleton and Umantsev did not characterise the exact obstruction to the Jamshidian decomposition, nor did they establish the minimax optimality of the projected strikes or derive an explicit quadratic error rate. Section 5 of the present paper provides these missing theoretical elements. *Brigo and Mercurio* (2006) provide a comprehensive treatment of affine short-rate models and their calibration to interest-rate derivative markets, including an exposition of the Jamshidian decomposition and its use in swaption pricing Brigo and Mercurio (2006). Their textbook documents the practical reliance on one-factor approximations in calibration routines for callable bonds and swaptions, contextualising the problem studied here.

3. Setup and Notation

3.1. State space and bond prices

Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ where $\mathbb{Q}$ is a risk-neutral pricing measure. Let $d \geq 1$ be the number of factors and let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Markov process.

Fix an option expiry $T_0 > 0$, payment dates $T_1 < \cdots < T_n$ with $T_i > T_0$, and positive cashflows $c_1, \ldots, c_n > 0$. Write $x := X_{T_0}$ and let $E \subset \mathbb{R}^d$ be the connected open domain of x.

Assumption 3.1 (Exponential-affine pricing)**.** For each $i = 1, \ldots, n$,

$$P_i(x) \;:=\; P(T_0, T_i;\, x) \;=\; A_i\, e^{-B_i^\top x}, \qquad x \in E, \tag{1}$$

where $A_i > 0$ and $B_i \in \mathbb{R}^d \setminus \{0\}$.

Remark 3.2 (Boundary issues and CIR-type models)**.** Assumption 3.1 requires E to be a connected open set. For Gaussian factor models (Vasicek, Hull–White), $E = \mathbb{R}^d$ and no boundary issues arise. For CIR-type models where each factor $X_{T_0}^{(k)}$ takes values in $(0, \infty)$, the natural state space $E = (0, \infty)^d$ is open, so the perturbation argument $x_\varepsilon = x^* + \varepsilon h \in E$ is valid for any $x^* \in E$ with ε chosen small enough that x_ε remains in $(0, \infty)^d$. The proof of Theorem 5.1 is therefore valid for CIR-type models provided the exercise-boundary point x^* lies in the interior of E, which is guaranteed since E is itself open.

3.2. The coupon bond and the decomposition problem

The *coupon bond value at expiry* is

$$C(x) := \sum_{i=1}^{n} c_i P_i(x).$$

Fix a strike $K > 0$.

Assumption 3.3 (Nontriviality)**.**

$$\inf_{x \in E} C(x) \; < \; K \; < \; \sup_{x \in E} C(x). \tag{NT}$$

Definition 3.4 (Exact Jamshidian decomposition)**.** An *exact Jamshidian decomposition* for strike K consists of deterministic strikes $\kappa_1, \ldots, \kappa_n > 0$ satisfying $\sum_i c_i \kappa_i = K$ and

$$\Big(C(x) - K\Big)^+ = \sum_{i=1}^{n} c_i \Big(P_i(x) - \kappa_i\Big)^+ \qquad \forall\, x \in E. \tag{J}$$

4. The Simultaneous Threshold Lemma

Lemma 4.1 (Simultaneous Threshold Lemma)**.** *Suppose* (J) *holds. Then for every $x \in E$, either $P_i(x) \geq \kappa_i$ for all i, or $P_i(x) \leq \kappa_i$ for all i. Moreover, if $C(x) = K$, then $P_i(x) = \kappa_i$ for all i.*

Proof. Case $C(x) > K$. Using $K = \sum_i c_i \kappa_i$ and (J):

$$\sum_i c_i (P_i(x) - \kappa_i)^+ = C(x) - K = \sum_i c_i (P_i(x) - \kappa_i).$$

Hence $\sum_i c_i (\kappa_i - P_i(x))^+ = 0$. Since every term is nonnegative and $c_i > 0$, each vanishes: $P_i(x) \geq \kappa_i$ for all i.
Case $C(x) < K$. The left-hand side of (J) is 0, so $\sum_i c_i (P_i(x) - \kappa_i)^+ = 0$, giving $P_i(x) \leq \kappa_i$ for all i.
Case $C(x) = K$. The same argument gives $P_i(x) \leq \kappa_i$ for all i. But $\sum_i c_i P_i(x) = K = \sum_i c_i \kappa_i$ with $c_i > 0$ and $P_i(x) \leq \kappa_i$ forces $P_i(x) = \kappa_i$ for all i. □

Remark 4.2. Geometrically, the lemma requires the shifted vector $(P_1(x)-\kappa_1, \ldots, P_n(x)-\kappa_n)$ to take values only in $\mathbb{R}^n_{\geq 0} \cup \mathbb{R}^n_{\leq 0}$ for every $x \in E$. This is a *pathwise comonotonicity* condition.

5. Exact Characterization

Theorem 5.1 (Exact obstruction to the Jamshidian decomposition)**.** *Let Assumptions 3.1 and 3.3 hold. The following are equivalent.*

(1) There exists an exact Jamshidian decomposition (J) *with deterministic strikes* $\kappa_i > 0$.

(2) There exist $u \in \mathbb{R}^d \setminus \{0\}$ *and* $b_i > 0$ *such that* $B_i = b_i u$ *for all* i.

Proof. **(2) ⇒ (1).** Set $Y(x) := u^\top x$. Then $P_i(x) = A_i e^{-b_i Y(x)}$, so every bond price is a strictly decreasing function of the same scalar Y. The function $g(y) := \sum_i c_i A_i e^{-b_i y}$ is continuous and strictly decreasing from $(0, \infty)$ onto $(0, \infty)$. By (NT) and the intermediate value theorem, there is a unique $y^* \in \mathbb{R}$ with $g(y^*) = K$. Set $\kappa_i := A_i e^{-b_i y^*}$; then $\sum_i c_i \kappa_i = K$. One verifies (J) case by case on $\{Y < y^*\}$, $\{Y > y^*\}$, $\{Y = y^*\}$.
(1) ⇒ (2). Assume (J) holds. Since E is connected and C is continuous, $C(E)$ is an interval. By (NT) and the IVT, $\exists\, x^* \in E$ with $C(x^*) = K$. Lemma 4.1 gives $P_i(x^*) = \kappa_i$ for all i.
Suppose for contradiction that B_i, B_j are not positively collinear for some i, j. Set $h := B_i / \|B_i\| - B_j / \|B_j\|$. By the strict Cauchy–Schwarz inequality:

$$B_i^\top h = \|B_i\| - \frac{B_i^\top B_j}{\|B_j\|} > 0, \qquad B_j^\top h = \frac{B_i^\top B_j}{\|B_i\|} - \|B_j\| < 0.$$

For small $\varepsilon > 0$, $x_\varepsilon := x^* + \varepsilon h \in E$ (since E is open), and

$$P_i(x_\varepsilon) = \kappa_i e^{-\varepsilon B_i^\top h} < \kappa_i, \qquad P_j(x_\varepsilon) = \kappa_j e^{-\varepsilon B_j^\top h} > \kappa_j.$$

This contradicts Lemma 4.1. Hence all B_i are pairwise positively collinear; by transitivity, $B_i = b_i u$ with $u = B_1 / \|B_1\|$ and $b_i = \|B_i\| > 0$. ☐

Corollary 5.2 (Impossibility in genuine multifactor models)**.** *If* $\{B_i\}_{i=1}^n$ *span a subspace of dimension at least* 2*, no exact deterministic-strike Jamshidian decomposition exists.*

6. Quantitative Near-Collinearity Theory

Theorem 5.1 is a sharp "all-or-nothing" result. In practice, loading vectors in a two- or three-factor model may be *nearly* collinear, especially at short maturities. This section quantifies how well the projected one-factor Jamshidian approximation performs when $\delta := \max_{i,j} \angle(B_i, B_j)$ is small.

6.1. Orthogonal decomposition and transverse dispersion

Fix a unit vector $u \in S^{d-1}$ satisfying

$$b_i := u^\top B_i > 0 \qquad (i = 1, \dots, n).$$

Write the orthogonal decomposition

$$B_i = b_i u + r_i, \qquad r_i := B_i - b_i u \;\in\; u^\perp.$$

For each state $x \in E$ decompose

$$x = yu + z, \qquad y := u^\top x, \qquad z := x - yu \;\in\; u^\perp.$$

Then

$$P_i(x) = A_i e^{-B_i^\top x} = A_i e^{-b_i y - r_i^\top z}.$$

For deterministic strikes $\kappa_i > 0$ with $\sum_i c_i \kappa_i = K$, define

$$\tau_i := \frac{1}{b_i} \log \frac{A_i}{\kappa_i}, \qquad \alpha_i := \frac{r_i}{b_i}, \qquad T_i(z) := \tau_i - \alpha_i^\top z. \tag{2}$$

Then

$$P_i(x) = \kappa_i \, e^{-b_i (y - T_i(z))},$$

so that

$$P_i(x) \ge \kappa_i \iff y \le T_i(z).$$

Definition 6.1 (Moving thresholds and transverse dispersion)**.** The *lower*, *upper thresholds* and *strip width* are

$$L_\kappa(z) := \min_i T_i(z), \qquad U_\kappa(z) := \max_i T_i(z), \qquad W_\kappa(z) := U_\kappa(z) - L_\kappa(z) \ge 0. \tag{3}$$

The *transverse dispersion* is

$$D_u := \max_{i,j} \left| \frac{r_i}{b_i} - \frac{r_j}{b_j} \right| = \max_{i,j} |\alpha_i - \alpha_j|. \tag{4}$$

The *deterministic-strike Jamshidian approximation* is

$$J_\kappa(x) := \sum_{i=1}^{n} c_i (P_i(x) - \kappa_i)^+.$$

Remark 6.2. $D_u = 0$ if and only if r_i / b_i is the same for all i, which by definition of r_i is equivalent to B_i being positively collinear with u. Hence $D_u = 0$ exactly in the setting where Theorem 5.1 guarantees the decomposition is exact.

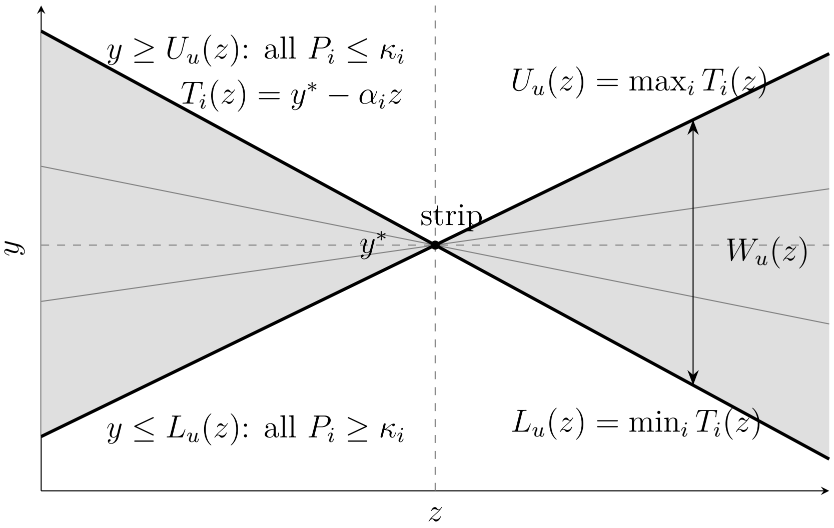

Figure 1: Illustration of strip localization under the projected strikes. For a one-dimensional transverse coordinate z, the individual moving thresholds are $T_i(z) = y^* - \alpha_i z$. The lower and upper envelopes $L_u(z) = \min_i T_i(z)$ and $U_u(z) = \max_i T_i(z)$ bound the strip where the deterministic-strike Jamshidian approximation may differ from the true payoff. Outside the strip, all bond terms are simultaneously above or below their respective thresholds.

6.2. Strip localization and pathwise error bound

Theorem 6.3 (Strip localization and pathwise upper bound)**.** *For every $x = yu+z \in E$,*

$$0 \;\leq\; J_\kappa(x) - \Big(C(x) - K\Big)^+ \;\leq\; \frac{K}{2}\Big(e^{b_{\max} W_\kappa(z)} - 1\Big)\, \mathbb{1}_{\{L_\kappa(z) \leq y \leq U_\kappa(z)\}}, \tag{5}$$

where $b_{\max} := \max_i b_i$. In particular, the approximation is exact outside the strip:

$$J_\kappa(x) = \Big(C(x) - K\Big)^+ \qquad \textit{whenever } y \leq L_\kappa(z) \textit{ or } y \geq U_\kappa(z).$$

Proof. Set $a_i := c_i(P_i(x) - \kappa_i)$. Then $\sum_i a_i = C(x) - K$, $J_\kappa(x) = \sum_i a_i^+$, $(C(x) - K)^+ = (\sum_i a_i)^+$. The lower bound $J_\kappa(x) \geq (C(x) - K)^+$ follows immediately from the subadditivity of $r \mapsto r^+$.

Exactness. Note $a_i \geq 0 \iff P_i(x) \geq \kappa_i \iff y \leq T_i(z)$. If $y \leq L_\kappa(z) = \min_i T_i(z)$, then $a_i \geq 0$ for all i, so $J_\kappa(x) = \sum_i a_i = C(x) - K = (C(x) - K)^+$. If $y \geq U_\kappa(z)$, then $a_i \leq 0$ for all i, so $J_\kappa(x) = 0 = (C(x) - K)^+$.

Quantitative bound. For any real numbers $a_1, \ldots, a_n$,

$$\sum_i a_i^+ - \Big(\sum_i a_i\Big)^+ \;\leq\; \min\Big\{\sum_i a_i^+,\, \sum_i a_i^-\Big\} \;\leq\; \tfrac{1}{2}\sum_i |a_i|.$$

Hence

$$0 \leq J_\kappa(x) - (C(x) - K)^+ \leq \tfrac{1}{2}\sum_i c_i \left|P_i(x) - \kappa_i\right|.$$

In the strip $L_\kappa(z) \le y \le U_\kappa(z)$, for every i: $|y - T_i(z)| \le W_\kappa(z)$, so

$$|P_i(x) - \kappa_i| = \kappa_i \Big| e^{-b_i(y - T_i(z))} - 1 \Big| \le \kappa_i \Big(e^{b_i W_\kappa(z)} - 1 \Big) \le \kappa_i \Big(e^{b_{\max} W_\kappa(z)} - 1 \Big).$$

Summing with $\sum_i c_i \kappa_i = K$ gives (5). $\square$

6.3. Projected strikes and minimax optimality

The *projected strikes* are defined by solving

$$\sum_{i=1}^{n} c_i A_i e^{-b_i y^*} = K \tag{6}$$

for the unique $y^* \in \mathbb{R}$ (existence and uniqueness follow since $y \mapsto \sum_i c_i A_i e^{-b_i y}$ is continuous, strictly decreasing, and surjective onto $(0, \infty)$), and setting

$$\kappa_i^u := A_i e^{-b_i y^*}. \tag{7}$$

Under the projected strikes, $\tau_i \equiv y^*$ for all i, so $T_i(z) = y^* - \alpha_i^\top z$ and

$$W_u(z) := W_{\kappa^u}(z) = \max_i(-\alpha_i^\top z) - \min_i(-\alpha_i^\top z) \le D_u \, \|z\| \, . \tag{8}$$

Proposition 6.4 (Minimax optimality of projected strikes)**.** *Fix u and $M > 0$. Among all deterministic strikes (κ_i) with $\sum_i c_i \kappa_i = K$, the projected strikes (κ_i^u) are the unique minimizers of the worst-case strip width $\sup_{\|z\| \le M} W_\kappa(z)$. The minimal value is $M D_u$.*

Remark 6.5. The assumption $\|f_{Y|Z}(\cdot|Z)\|_\infty \le L$ a.s. is satisfied in all standard affine Gaussian models. In the Vasicek and Hull–White frameworks, $(Y, Z) = (u^\top X_{T_0}, X_{T_0} - Y u)$ is jointly Gaussian under Q^{T_0}; the conditional distribution of Y given Z is then Gaussian with variance $\sigma_Y^2 := \mathrm{Var}^{T_0}(Y) - \mathrm{Cov}^{T_0}(Y, Z)\mathrm{Var}^{T_0}(Z)^{-1}\mathrm{Cov}^{T_0}(Z, Y)$, which is bounded away from zero by model parameters, so the conditional density is bounded by $L = (2\pi\sigma_Y^2)^{-1/2} < \infty$. The assumption also holds whenever Y and Z are independent under Q^{T_0} and Y has a bounded marginal density, which covers the independence structure of Example 6.9.

Proof. Since $T_i(z) = \tau_i - \alpha_i^\top z$,

$$W_\kappa(z) = \max_i(\tau_i - \alpha_i^\top z) - \min_j(\tau_j - \alpha_j^\top z) = \max_{i,j}\Big[(\tau_i - \tau_j) - (\alpha_i - \alpha_j)^\top z\Big].$$

Taking the sup over $\|z\| \le M$:

$$\sup_{\|z\| \le M} W_\kappa(z) = \max_{i,j}\Big[|\tau_i - \tau_j| + M|\alpha_i - \alpha_j|\Big].$$

This is at least $M \max_{i,j} |\alpha_i - \alpha_j| = M D_u$, with equality if and only if $\tau_i = \tau_j$ for all i, j. Among choices satisfying $\sum_i c_i \kappa_i = K$, the constraint $\tau_i = \tau_j \equiv \tau$ for all i forces $\sum_i c_i A_i e^{-b_i \tau} = K$, which has the unique solution $\tau = y^*$. $\square$

Remark 6.6. Proposition 6.4 gives a robust, distribution-free optimality result. The projected strikes are the unique best deterministic strikes in the minimax sense: they are optimal regardless of the distribution of the transverse component $Z = X_{T_0} - Yu$.

6.4. Pricing error under the projected strikes

Under the T_0-forward measure Q^{T_0},

$$V = P(0,T_0)\,\mathbb{E}^{T_0}\Big[(C(X_{T_0}) - K)^+\Big], \qquad V_u = P(0,T_0)\,\mathbb{E}^{T_0}\Big[J_u(X_{T_0})\Big].$$

Theorem 6.7 (Quadratic pricing error bound)**.** *Assume that under Q^{T_0}, the conditional law of $Y := u^\top X_{T_0}$ given $Z := X_{T_0} - Yu$ admits an almost-surely bounded density:*

$$\|f_{Y|Z}(\,\cdot\,|Z)\|_\infty \le L \quad a.s.$$

Then

$$0 \;\le\; V_u - V \;\le\; \frac{P(0,T_0)\,K\,L}{2}\,\mathbb{E}^{T_0}\Big[W_u(Z)\Big(e^{b_{\max}W_u(Z)} - 1\Big)\Big].$$

If additionally $D_u \le \bar{D}$ and $\mathbb{E}^{T_0}\Big[\|Z\|^2\,e^{b_{\max}\bar{D}\|Z\|}\Big] < \infty$, then

$$0 \;\le\; V_u - V \;\le\; \frac{P(0,T_0)\,K\,L\,b_{\max}}{2}\,D_u^2\,\mathbb{E}^{T_0}\Big[\|Z\|^2\,e^{b_{\max}\bar{D}\|Z\|}\Big]. \tag{9}$$

In particular,

$$V_u - V \;=\; O(D_u^2) \quad \text{as } D_u \downarrow 0,$$

which translates to $O(\delta^2)$ as $\delta = \max_{i,j}\angle(B_i, B_j) \downarrow 0$.

Remark 6.8 (On the moment condition)**.** The integrability requirement $\mathbb{E}^{T_0}\Big[\|Z\|^2\,e^{b_{\max}\bar{D}\|Z\|}\Big] < \infty$ is mild in Gaussian models: if Z is Gaussian under Q^{T_0}, then $\|Z\|$ has sub-Gaussian tails and the exponential moment is finite for any fixed $\bar{D} < \infty$. The condition may fail for heavy-tailed factor distributions or if $\bar{D}$ is large (i.e. when the loading vectors are far from collinear), but in the near-collinearity regime $D_u \le \bar{D} \ll 1$, it is automatically satisfied for all models in the affine Gaussian class. When the moment condition fails, the first inequality in Theorem 6.7—the $W_u(Z)(e^{b_{\max}W_u(Z)} - 1)$ bound—remains valid; only the $O(D_u^2)$ refinement requires the additional integrability.

Proof. From Theorem 6.3,

$$0 \le J_u(X_{T_0}) - (C(X_{T_0}) - K)^+ \le \tfrac{K}{2}\Big(e^{b_{\max}W_u(Z)} - 1\Big)\mathbb{1}_{\{Y \in [L_u(Z), U_u(Z)]\}}.$$

Taking Q^{T_0}-expectations and conditioning on Z:

$$0 \le V_u - V \le \frac{P(0,T_0)K}{2}\,\mathbb{E}^{T_0}\Big[\Big(e^{b_{\max}W_u(Z)} - 1\Big)Q^{T_0}\Big(Y \in [L_u(Z), U_u(Z)]\,\Big|\,Z\Big)\Big].$$

The density bound gives $Q^{T_0}(Y \in [L_u(Z), U_u(Z)]\,|\,Z) \le L \cdot W_u(Z)$, establishing the first inequality.

For (9), use $W_u(Z) \le D_u \|Z\|$ (from (8)) and the elementary inequality $e^a - 1 \le ae^a$ for $a \ge 0$:

$$W_u(Z)\Big(e^{b_{\max} W_u(Z)} - 1\Big) \le b_{\max} W_u(Z)^2 e^{b_{\max} W_u(Z)} \le b_{\max} D_u^2 \|Z\|^2 e^{b_{\max} \bar{D}\|Z\|}.$$

Taking expectations completes the proof. □

6.5. Sharpness of the quadratic rate

Example 6.9 (Sharpness)**.** Take $d = 2$, $u = e_1$, $v = e_2$, $n = 2$, $c_1 = c_2 = A_1 = A_2 = 1$,

$$B_1 = bu, \qquad B_2^\varepsilon = bu + \varepsilon v, \qquad b > 0.$$

Then $D_u = \varepsilon/b$ and $\angle(B_1, B_2^\varepsilon) = \arctan(\varepsilon/b) = \varepsilon/b + O(\varepsilon^2)$. Let $K = 2e^{-by^*}$, so $\kappa_1^u = \kappa_2^u = \kappa := e^{-by^*}$. Assume under Q^{T_0} that $Y = e_1^\top X_{T_0}$ and $Z = e_2^\top X_{T_0}$ are independent, Y has a continuous density f_Y with $f_Y(y^*) > 0$, and $\mathbb{E}^{T_0}[Z^2] < \infty$. Then as $\varepsilon \downarrow 0$,

$$V_\varepsilon^u - V_\varepsilon \;=\; \frac{P(0,T_0)\,\kappa\, f_Y(y^*)}{4b}\, \varepsilon^2\, \mathbb{E}^{T_0}[Z^2] \;+\; o(\varepsilon^2).$$

Since $\delta = \varepsilon/b + O(\varepsilon^2)$, the pricing error is genuinely of order δ^2.

Proof sketch. Under the independence of Y and Z and the projected strikes $\kappa_1^u = \kappa_2^u = \kappa = e^{-by^*}$, one has

$$T_1(z) = y^*, \qquad T_2^\varepsilon(z) = y^* - \tfrac{\varepsilon}{b}\, z,$$

so the strip $[L_u(z), U_u(z)]$ equals $[\min(y^*, y^* - \frac{\varepsilon}{b}z), \max(y^*, y^* - \frac{\varepsilon}{b}z)]$, which has width $W_u(z) = \frac{\varepsilon}{b}|z|$.

The pricing error equals

$$V_\varepsilon^u - V_\varepsilon = P(0,T_0)\, \mathbb{E}^{T_0}\Big[J_u(X_{T_0}) - (C(X_{T_0}) - K)^+\Big].$$

Inside the strip, the mismatch arises from the bond P_2^ε having a threshold displaced from y^* by $-\frac{\varepsilon}{b}Z$. Expanding to second order in ε and using the independence of Y and Z:

$$J_u(X_{T_0}) - (C(X_{T_0}) - K)^+ = \kappa \left| e^{-\varepsilon Z/b} - 1 \right| \mathbb{1}_{\{Y \text{ between } y^* \text{ and } y^* - \frac{\varepsilon}{b}Z\}} + O(\varepsilon^3).$$

The indicator event has Q^{T_0}-probability $\approx f_Y(y^*) \cdot \frac{\varepsilon}{b}|Z|$ by the density of Y at y^*, and $|e^{-\varepsilon Z/b} - 1| = \frac{\varepsilon}{b}|Z| + O(\varepsilon^2)$. Taking expectations:

$$\mathbb{E}^{T_0}\Big[J_u - (C-K)^+\Big] = \kappa\, f_Y(y^*)\, \frac{\varepsilon^2}{b^2}\, \mathbb{E}^{T_0}[Z^2] \cdot \tfrac{1}{2}\, b + o(\varepsilon^2) = \frac{\kappa\, f_Y(y^*)}{4b}\, \varepsilon^2\, \mathbb{E}^{T_0}[Z^2] + o(\varepsilon^2),$$

where the factor $\frac{1}{2}$ arises from integrating the indicator over the strip. Multiplying by $P(0,T_0)$ gives the stated asymptotic. □ □

6.6. Angular-spread form of the bound

Since $|\alpha_i| = \|r_i\| / b_i = \tan \angle(B_i, u)$, we have

$$D_u \le 2\tan(\theta_u), \qquad \theta_u := \max_i \angle(B_i, u).$$

With $u = B_1 / \|B_1\|$ and $\delta = \max_{i,j} \angle(B_i, B_j)$, one has $\theta_u \le \delta$. Theorem 6.7 then gives, for any $\delta_0 < \pi/2$ and $\delta \le \delta_0$:

$$0 \le V_u - V \le 2\,P(0,T_0)\,K\,L\,b_{\max}\,\tan^2\!\delta\;\mathbb{E}^{T_0}\Big[\|Z\|^2\,e^{2b_{\max}\tan(\delta_0)\|Z\|}\Big]. \tag{10}$$

Therefore $V_u - V = O(\delta^2)$ as $\delta \downarrow 0$, with a constant that is explicit and finite under the moment condition in Theorem 6.7.

7. Adaptive-Strike Triviality and Numéraire Invariance

A natural response to the exact obstruction in Theorem 5.1 of the main paper is to generalize the notion of a Jamshidian decomposition: allow the component strikes κ_i to depend on the state x rather than being deterministic constants. The results below show that this generalization is either trivially permissive (unrestricted state-dependence) or structurally equivalent to the original one-factor constraint (scalar-statistic dependence).

7.1. State-Dependent-Strike Triviality

Proposition 7.1 (State-dependent-strike triviality). *For any positive component prices $P_i(x)$, positive cashflows c_i, and strike $K > 0$, define*

$$C(x) := \sum_{i=1}^{n} c_i P_i(x), \qquad \kappa_i(x) := \frac{K\,P_i(x)}{C(x)}.$$

Then $\sum_i c_i\,\kappa_i(x) = K$ for every $x \in E$, and the pathwise identity

$$\Big(C(x) - K\Big)^+ = \sum_{i=1}^{n} c_i\Big(P_i(x) - \kappa_i(x)\Big)^+ \tag{11}$$

holds for every $x \in E$.

Proof. The strike-sum identity is immediate: $\sum_i c_i\,\kappa_i(x) = (K/C(x))\sum_i c_i P_i(x) = K$.
Case $C(x) > K$. For each i,

$$P_i(x) - \kappa_i(x) = P_i(x)\left(1 - \frac{K}{C(x)}\right) \ge 0,$$

so every term on the right of (11) is non-negative and

$$\sum_i c_i\Big(P_i(x) - \kappa_i(x)\Big)^+ = \left(1 - \frac{K}{C(x)}\right)\sum_i c_i P_i(x) = C(x) - K.$$

Case $C(x) \le K$. Then $1 - K/C(x) \le 0$, so $P_i(x) - \kappa_i(x) \le 0$ for every i, each positive part vanishes, and the right side equals $0 = (C(x) - K)^+$. ☐

Remark 7.2 (Numéraire invariance of κ_i)**.** The proportional allocation $\kappa_i(x) = KP_i(x)/C(x)$ is invariant under any strictly positive numéraire $N(x) > 0$:

$$\frac{\kappa_i(x)}{N(x)} = \frac{K\Big(P_i(x)/N(x)\Big)}{\Big(C(x)/N(x)\Big)}.$$

Hence restricting attention to numéraire-invariant state-dependent strikes does not reduce the scope of Theorem 7.1: the proportional construction remains available after any change of numéraire.

The proposition implies that the question "does an adaptive Jamshidian decomposition exist?" has a trivially affirmative answer with no structural content. Any meaningful refinement must impose an additional restriction on the form of κ_i.

7.2. Numéraire Changes Do Not Alter Exercise Geometry

Lemma 7.3 (Numéraire invariance of the exercise set)**.** *For any strictly positive numéraire* $N : E \to (0, \infty)$*,*

$$\Big(C(x) - K\Big)^+ = N(x)\left(\frac{C(x)}{N(x)} - \frac{K}{N(x)}\right)^+,$$

and the exercise set is unchanged:

$$\{x \in E : C(x) > K\} = \Big\{x \in E : \tfrac{C(x)}{N(x)} > \tfrac{K}{N(x)}\Big\}.$$

Proof. Since $N(x) > 0$, the map $t \mapsto N(x)\, t^+$ satisfies $N(x)\,(t/N(x))^+ = t^+$ for all $t \in \mathbb{R}$. Apply with $t = C(x) - K$:

$$N(x)\Big(\frac{C(x) - K}{N(x)}\Big)^+ = (C(x) - K)^+.$$

The set identity follows because $C(x) > K \iff C(x)/N(x) > K/N(x)$ (as $N(x) > 0$). ☐

Corollary 7.4 (Numéraire changes cannot resolve multifactor geometry)**.** *If the loading vectors* $\{B_i\}_{i=1}^n$ *span a subspace of dimension* ≥ 2 *in* $\mathbb{R}^d$*, then for* every *strictly positive numéraire* N *and every collection of deterministic component strikes* $\kappa_i > 0$ *with* $\sum_i c_i\kappa_i = K$*, the identity*

$$\Big(C(x) - K\Big)^+ = \sum_i c_i\Big(P_i(x) - \kappa_i\Big)^+$$

fails on a set of positive Lebesgue measure in E*.*

Proof. By Theorem 7.3, changing the numéraire leaves the exercise set $\{C > K\}$ invariant. Theorem 5.1 of the main paper establishes that, when $\{B_i\}$ span dimension ≥ 2, no deterministic-strike decomposition exists for the original coupon bond (C, K). Because the exercise set is unchanged by the numéraire change, the same obstruction applies to $(C/N,\ K/N)$ with any positive numéraire N. ☐

Remark 7.5. Theorem 7.4 confirms that the collinearity obstruction is genuinely geometric: it lives in the factor space E, not in the choice of probability measure or pricing units. The standard change-of-numéraire technique of Geman, El Karoui and Rochet (1995) is, of course, invaluable for simplifying option pricing in other respects; the present corollary simply records that it cannot convert a multifactor exercise boundary into a one-dimensional one.

8. Scalar-Factor Rigidity

The exact characterization theorem of the main paper establishes that a *deterministic-strike* Jamshidian decomposition holds if and only if all B_i are positively collinear. A natural strengthening asks whether some *nonlinear* scalar reduction—allowing $P_i = f_i \circ g$ for a possibly nonlinear Borel function g—could restore comonotonicity in a multifactor setting. The following theorem shows that the answer is no: for exponential-affine families, positive collinearity is necessary and sufficient for *any* scalar-factor reduction.

Theorem 8.1 (Scalar-factor rigidity / comonotonicity classification)**.** *Let $\Phi_i(x) = A_i e^{B_i^\top x}$ on a connected open set $E \subseteq \mathbb{R}^d$, with $A_i > 0$ and $B_i \neq 0$. The following are equivalent.*

(i) *There exist a continuous surjective map $g\colon E \to I \subseteq \mathbb{R}$ and strictly monotone (all in the same direction) continuous functions $f_i\colon I \to (0, \infty)$ such that $\Phi_i = f_i \circ g$ for all i.*

(ii) *The family $\{\Phi_i\}$ is pairwise pathwise comonotone:*

$$\Big(\Phi_i(x) - \Phi_i(y)\Big)\Big(\Phi_j(x) - \Phi_j(y)\Big) \geq 0 \qquad \forall\, x, y \in E,\ \forall\, i, j.$$

(iii) *The loading vectors are positively collinear: $B_i = b_i u$ for some $u \in \mathbb{R}^d \setminus \{0\}$ and $b_i > 0$, for all $i = 1, \ldots, n$.*

Proof. We prove the cycle (iii) ⇒ (i) ⇒ (ii) ⇒ (iii).

(iii) ⇒ (i). Set $g(x) := u^\top x$, which is continuous and surjective from the connected open set E onto an open interval $I \subseteq \mathbb{R}$. Define $f_i(t) := A_i e^{b_i t}$, which is strictly increasing in t for each i (since $b_i > 0$). Then

$$\Phi_i(x) \;=\; A_i e^{B_i^\top x} \;=\; A_i e^{b_i u^\top x} \;=\; f_i(g(x)) \qquad \forall\, x \in E,\ \forall\, i,$$

so $\Phi_i = f_i \circ g$ with g and each f_i monotone in the same (increasing) direction.

(i) ⇒ (ii). Fix any $x, y \in E$ and any pair i, j. Since f_i and f_j are both monotone in the same direction and share the scalar g, either $g(x) \geq g(y)$—in which case both $\Phi_i(x) - \Phi_i(y) \geq 0$ and $\Phi_j(x) - \Phi_j(y) \geq 0$—or $g(x) \leq g(y)$—in which case both differences are ≤ 0. In either case their product is ≥ 0.

(ii) ⇒ (iii). We show that if B_i and B_j are *not* positively collinear for some pair i, j, then (ii) is violated.

Constructing the perturbation. Suppose $B_i \neq \lambda B_j$ for any $\lambda > 0$. Set

$$v \;:=\; \frac{B_i}{\|B_i\|} - \frac{B_j}{\|B_j\|}.$$

By the strict Cauchy–Schwarz inequality applied to the non-collinear pair (B_i, B_j),

$$B_i^\top v \;=\; \|B_i\| - \frac{B_i^\top B_j}{\|B_j\|} \;>\; 0, \qquad B_j^\top v \;=\; \frac{B_i^\top B_j}{\|B_i\|} - \|B_j\| \;<\; 0.$$

Violating comonotonicity. Fix any $x^* \in E$. Since E is open, there exists $\varepsilon_0 > 0$ such that $x^* + \varepsilon v \in E$ for all $\varepsilon \in (0, \varepsilon_0)$. Set $x := x^* + \varepsilon v$ and $y := x^*$. Then

$$\begin{aligned} \Phi_i(x) - \Phi_i(y) \;&=\; \Phi_i(x^*)\Big(e^{\varepsilon B_i^\top v} - 1\Big) \;>\; 0, \\ \Phi_j(x) - \Phi_j(y) \;&=\; \Phi_j(x^*)\Big(e^{\varepsilon B_j^\top v} - 1\Big) \;<\; 0, \end{aligned}$$

since $B_i^\top v > 0$, $B_j^\top v < 0$, and $\Phi_i(x^*), \Phi_j(x^*) > 0$. Hence

$$\Big(\Phi_i(x) - \Phi_i(y)\Big)\Big(\Phi_j(x) - \Phi_j(y)\Big) \;<\; 0,$$

contradicting (ii). Therefore every pair B_i, B_j is positively collinear. By transitivity, $B_i = b_i u$ for a common direction $u \in \mathbb{R}^d \setminus \{0\}$. The requirement $B_i^\top v > 0$ forces $b_i\, u^\top v > 0$, so $b_i > 0$ once u is fixed with $u^\top v > 0$. □

Corollary 8.2 (Nonlinear scalar factors are no more powerful than linear ones)**.** *For an exponential-affine family $\{\Phi_i\}$, any nonlinear scalar-factor reduction $\Phi_i = f_i \circ g$ is equivalent—in terms of what it requires of the loadings B_i—to the linear projection $g(x) = u^\top x$. No exotic choice of g can restore comonotonicity when the B_i fail to be positively collinear.*

Remark 8.3 (Measurability)**.** Theorem 8.1 is stated for *continuous* g and f_i on a connected domain. If g is merely Borel measurable, condition **??** still forces **??** via the same perturbation argument, since the perturbation operates on open sets where measurability issues are irrelevant.

9. Discussion

9.1. Geometric interpretation

The map $x \mapsto (P_1(x), \ldots, P_n(x))$ traces a surface $\mathcal{S}$ in $\mathbb{R}^n_+$. The Jamshidian identity requires the hyperplane $\{\sum_i c_i z_i = K\}$ to separate $\mathcal{S}$ into two orthants: the region where all $z_i \geq \kappa_i$ and the region where all $z_i \leq \kappa_i$. When all $B_i = b_i u$, $\mathcal{S}$ is a one-dimensional curve and this separation is achievable. When the B_i span dimension ≥ 2, the perturbation h in the proof of Theorem 5.1 provides an explicit direction along which some bond prices rise while others fall.

9.2. Open problems

Several natural directions remain open. First, while Proposition 6.4 establishes that the projected strikes κ_i^u are minimax-optimal over the worst-case transverse state z, it is an open question whether they also minimise the expected pricing error under a specific fixed distribution of (Y, Z). Second, the bound (10) depends on the choice of reference direction u, and minimising it over $u \in \mathbb{S}^{d-1}$ defines a finite-dimensional optimisation problem whose solution, presumably a principal direction of the geometry of $\{B_i\}$, remains uncharacterised. Third, Theorem 5.1 rules out only the classical deterministic-strike form of the decomposition; whether exact decompositions of a more general kind allowing state-dependent component strikes or exploiting a change-of-numéraire to reduce the exercise set to a one-dimensional threshold can be constructed in genuinely multifactor models is an intriguing open problem. Finally, the exponential-affine assumption $P_i(x) = A_i e^{-B_i^\top x}$ is essential to the collinearity characterisation, and extending the obstruction theorem to quadratic Gaussian models, where $P_i(x) = A_i e^{-x^\top Q_i x - B_i^\top x}$, presents a natural and technically demanding next step.

10. Conclusion

We have established that within exponential-affine term structure models, the classical Jamshidian deterministic-strike decomposition holds if and only if all payment-date loading vectors are positively collinear (Theorem 5.1). This sharpens the folk understanding that the trick relies on one-factor structure to a precise and verifiable algebraic condition.In the near-collinear case, we have shown that the projected one-factor strikes are minimax-optimal (Proposition 6.4), the approximation error is confined to a strip around the projected exercise hyperplane (Theorem 6.3), and the pricing error is $O(\delta^2)$ in the angular spread—a bound that is sharp (Example 6.9) and carries an explicit constant (Theorem 6.7). Together, these results provide a complete and rigorous foundation for the Jamshidian approximation in modern multifactor interest rate models.

Acknowledgements. The author thanks the research community at MAHE Bengaluru for a stimulating intellectual environment.

References

Brigo, D. and Mercurio, F. (2006). *Interest Rate Models—Theory and Practice*, 2nd ed. Springer Finance, Berlin.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407.

Dai, Q. and Singleton, K. J. (2000). Specification analysis of affine term structure models. *Journal of Finance*, 55(5):1943–1978.

Duffie, D. and Kan, R. (1996). A yield-factor model of interest rates. *Mathematical Finance*, 6(4):379–406.

Duffie, D., Filipović, D., and Schachermayer, W. (2003). Affine processes and applications in finance. *Annals of Applied Probability*, 13(3):984–1053.

Frenk, J. B. G. and Kassay, G. (2006). On noncooperative games, minimax theorems and equilibrium problems. In *Pareto Optimality, Game Theory and Equilibria*, Springer Optimization and Its Applications, vol. 17, pp. 53–93.

Geman, H., El Karoui, N., and Rochet, J.-C. (1995). Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability*, 32(2):443–458.

Hull, J. and White, A. (1990). Pricing interest-rate-derivative securities. *Review of Financial Studies*, 3(4):573–592.

Jamshidian, F. (1989). An exact bond option formula. *Journal of Finance*, 44(1):205–209.

Pelsser, A. (2006). Pricing swaptions and coupon bond options in affine term structure models. *Mathematical Finance*, 16(4):673–694.

Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill, New York.

Rudin, W. (1991). *Functional Analysis*, 2nd ed. McGraw-Hill, New York.

Singleton, K. J. and Umantsev, L. (2002). Pricing coupon-bond options and swaptions in affine term structure models. *Mathematical Finance*, 12(4):427–446.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188.

Wei, J. Z. (1997). A simple approach to bond option pricing. *Journal of Futures Markets*, 17(2):131–160.