The Exact Obstruction to Jamshidian's Decomposition in Multifactor Affine Models

Aryan Ayyar

Manipal Academy of Higher Education

Bengaluru, India

ayyar.aryan@manipal.edu

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Abstract

Jamshidian's decomposition writes a European option on a coupon bond as a portfolio of European options on the constituent zero-coupon bonds, with deterministic component strikes obtained from a single scalar equation. We determine exactly when such a decomposition can hold in the exponential-affine class $P_i(x) = A_i e^{-B_i^\top x}$: a deterministic-strike decomposition exists if and only if the loading vectors $B_1, \ldots, B_n$ lie on a common positive ray in $\mathbb{R}^d$. When the loadings are only approximately collinear, we show that the natural projected one-factor strikes produce a pathwise upper bound for the payoff whose error vanishes outside a strip around a projected exercise hyperplane, derive an upper bound on the induced pricing error in terms of a transverse dispersion coefficient, and prove that the projected strikes are minimax-optimal for worst-case strip width. A worked two-factor Gaussian example illustrates the geometry and the size of the bound. Finally, we show that the obstruction is robust: unrestricted state-dependent strikes make the decomposition problem vacuous, changes of numéraire leave the exercise geometry unchanged, and no scalar reparameterization of the factor space, linear or otherwise, can restore the decomposition when the loadings fail to be collinear.

Keywords: coupon bond options, Jamshidian decomposition, affine term structure models, comonotonicity, swaptions.

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1. Introduction

A European option on a coupon bond, equivalently a European swaption under deterministic tenor conventions, has payoff $\left(\sum_{i=1}^n c_i P(T_0, T_i) - K\right)^+$ at its expiry T_0, where $P(T_0, T_i)$ are the prices at T_0 of the zero-coupon bonds maturing at the payment dates $T_1 < \cdots < T_n$. Jamshidian [9] observed that in a one-factor model this payoff splits exactly. If every bond price at expiry is a strictly decreasing function of the same scalar state variable r, one solves $\sum_i c_i P_i(r^*) = K$ for the critical state r^*, sets $\kappa_i := P_i(r^*)$, and obtains the pathwise identity

$$\Big(\sum_i c_i P_i(r) - K\Big)^+ \;=\; \sum_i c_i \Big(P_i(r) - \kappa_i\Big)^+.$$

The option on the portfolio becomes a portfolio of options, each of which has a closed form in the classical one-factor models of Vasicek [13], Cox, Ingersoll and Ross [3], and Hull and White [8].

It is well known that this device fails in genuinely multifactor models, and a substantial literature has developed accurate approximations for that case; see Wei [14], Munk [10], Singleton and Umantsev [12], Collin-Dufresne and Goldstein [2], and Schrager and Pelsser [11]. What the literature does not make explicit is the exact boundary of validity: within a given class of models, precisely which configurations admit the decomposition, and which do not? The usual statement, that the trick "requires one factor," is a sufficient condition phrased as folklore, not a characterization.

This paper closes that gap for the exponential-affine class, in which each bond price at expiry takes the form $P_i(x) = A_i e^{-B_i^\top x}$ for a state vector $x \in \mathbb{R}^d$. This class contains the models of Vasicek–Hull–White and CIR type and, more generally, the affine framework of Duffie and Kan [6].

The paper makes three contributions. First, an exact characterization (Theorem 4.1): a deterministic-strike Jamshidian decomposition exists if and only if the loading vectors $B_1, \dots, B_n$ are positively collinear, that is, lie on a single ray from the origin. The proof has two ingredients. A Simultaneous Threshold Lemma (Lemma 3.1) converts the pathwise payoff identity into a sign-coherence condition: at every state, all bond prices must sit on the same side of their component strikes. A perturbation argument then shows that any two non-collinear loadings admit a direction in factor space along which one bond price rises while the other falls, destroying coherence near the exercise boundary.

Second, a quantitative theory for the near-collinear regime (Section 5). Loadings calibrated to market data are typically close to, but not exactly on, a common ray. For a reference direction u we define a transverse dispersion coefficient D_u that vanishes precisely in the collinear case, and we show: the projected one-factor strikes make the decomposition a pathwise upper bound whose error is confined to a strip around a projected exercise hyperplane (Theorem 5.3); the projected strikes minimize the worst-case strip width among all deterministic-strike vectors (Proposition 5.4); and, under a bounded-conditional-density assumption, the pricing error is at most of order D_u^2 (Theorem 5.6). These are upper bounds and a worst-case optimality statement; we do not claim a converse relating small pricing error to near-collinearity. A worked two-factor Gaussian example (Section 6) reports the size of every quantity in the bound and compares the bound with the realized error.

Third, a robustness analysis of the obstruction itself (Section 7). Allowing the component strikes to depend on the state in an unrestricted way makes the decomposition problem vacuous: an explicit proportional-allocation rule always works, for any model whatsoever (Proposition 7.1). Changes of numéraire leave both the payoff identity and the exercise region unchanged, so the obstruction is geometric rather than measure-theoretic (Section 7.2). Finally, a rigidity theorem (Theorem 7.5) shows that for exponential-affine families, scalar-factor representability through *any* continuous monotone link functions, pairwise pathwise comonotonicity, and positive collinearity of the loadings are all equivalent. No nonlinear reparameterization of the state space can substitute for collinearity.

1.1. Related literature

The affine framework of Duffie and Kan [6] supplies the exponential-affine bond-price form used throughout; the classification of Dai and Singleton [4] shows that in canonical multifactor specifications non-collinear loadings are the generic case, so the failure identified here is structural rather than accidental. The foundations of regular affine processes are given by Duffie, Filipović and Schachermayer [5]. On the pricing side, Wei [14] and Munk [10] approximate a coupon bond by a single zero-coupon bond with matched sensitivity ("stochastic duration"); Singleton and Umantsev [12] approximate the multifactor exercise boundary by a hyperplane; Collin-Dufresne and Goldstein [2] use an Edgeworth expansion of the coupon-bond price distribution; Schrager and Pelsser [11] derive approximate affine swap-rate dynamics. All of these are approximation methods that presuppose the failure of the exact decomposition. The present paper is complementary: it identifies the exact boundary between decomposable and non-decomposable configurations, and quantifies the cost of projecting onto a one-factor geometry when the configuration is near that boundary. Textbook treatments of the one-factor trick can be found in Brigo and Mercurio [1].

1.2. Organization

Section 2 fixes the setting and defines the decomposition problem. Section 3 proves the Simultaneous Threshold Lemma. Section 4 proves the characterization theorem. Section 5 develops the near-collinearity theory, and Section 6 contains the worked example. Section 7 treats state-dependent strikes, numéraire changes, and scalar-factor rigidity. Section 8 discusses the geometry of the result and lists open problems.

2. Setup and the Decomposition Problem

2.1. State space and bond prices

Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ carrying an $\mathbb{R}^d$-valued Markov process $X = (X_t)_{t\geq 0}$, $d \geq 1$, where $\mathbb{Q}$ is a risk-neutral pricing measure. Fix an option expiry $T_0 > 0$, payment dates $T_1 < \cdots < T_n$ with $T_i > T_0$, and cashflows $c_1, \ldots, c_n > 0$. Write $x := X_{T_0}$ and let $E \subseteq \mathbb{R}^d$ be a connected open set containing the support of x; we refer to E as the state domain.

Assumption 2.1 (Exponential-affine pricing)**.** For each $i = 1, \ldots, n$, the time-T_0 price of the zero-coupon bond maturing at T_i is

$$P_i(x) \;:=\; P(T_0, T_i; x) \;=\; A_i\, e^{-B_i^\top x}, \qquad x \in E, \tag{1}$$

where $A_i > 0$ and $B_i \in \mathbb{R}^d \setminus \{0\}$.

Remark 2.2 (State domains of standard models)**.** For Gaussian factor models (Vasicek, Hull–White, multifactor G2-type extensions), $E = \mathbb{R}^d$. For CIR-type models each factor takes values in $(0, \infty)$ and $E = (0, \infty)^d$, which is again connected and open. All arguments below use only that E is connected and open; in particular the perturbation in the proof of Theorem 4.1 stays inside E for sufficiently small step size, regardless of the boundary behaviour of the factor process.

2.2. The decomposition problem

The value of the coupon-bearing bond at expiry is

$$C(x) := \sum_{i=1}^{n} c_i P_i(x), \qquad x \in E.$$

Fix a strike $K > 0$. To exclude options that are never, or always, exercised we impose:

Assumption 2.3 (Nontriviality)**.**

$$\inf_{x\in E} C(x) \;<\; K \;<\; \sup_{x\in E} C(x). \tag{NT}$$

Definition 2.4 (Exact Jamshidian decomposition)**.** An *exact Jamshidian decomposition* for strike K consists of deterministic strikes $\kappa_1, \ldots, \kappa_n > 0$ with $\sum_{i=1}^n c_i \kappa_i = K$ such that

$$\Big(C(x) - K\Big)^+ \;=\; \sum_{i=1}^{n} c_i \Big(P_i(x) - \kappa_i\Big)^+ \qquad \text{for all } x \in E. \tag{J}$$

Remark 2.5. The strike-consistency constraint $\sum_i c_i \kappa_i = K$ is part of Jamshidian's classical construction and is automatic whenever the option can be deep in the money: if there is a state at which $P_i(x) \geq \kappa_i$ for all i and $C(x) > K$, then (J) forces

$C(x) - K = \sum_i c_i(P_i(x) - \kappa_i)$, hence $\sum_i c_i\kappa_i = K$. We include it in the definition so that (J) is a single pathwise identity with a well-defined strike budget.

The practical content of Definition 2.4 is that each summand on the right of (J) is a European option on a single zero-coupon bond, which has a closed-form price in the standard affine Gaussian and CIR models. The question is when (J) can hold *pathwise*, as an identity between functions on E.

3. The Simultaneous Threshold Lemma

The pathwise identity (J) has a strong algebraic consequence: at every state, the shifted price vector $(P_1(x) - \kappa_1, \ldots, P_n(x) - \kappa_n)$ cannot have components of mixed sign. This is the only consequence of (J) that the necessity direction of Theorem 4.1 uses.

Lemma 3.1 (Simultaneous Threshold Lemma)**.** *Let $\kappa_1, \ldots, \kappa_n > 0$ satisfy $\sum_i c_i\kappa_i = K$ and* (J)*. Then for every $x \in E$, either $P_i(x) \geq \kappa_i$ for all i, or $P_i(x) \leq \kappa_i$ for all i. Moreover, if $C(x) = K$, then $P_i(x) = \kappa_i$ for all i.*

Proof. Fix $x \in E$ and suppose first that $C(x) \leq K$. Then $(C(x) - K)^+ = 0$, so (J) gives

$$\sum_{i=1}^{n} c_i\Big(P_i(x) - \kappa_i\Big)^+ = 0.$$

Every summand is nonnegative and every c_i is strictly positive, so $(P_i(x) - \kappa_i)^+ = 0$, that is, $P_i(x) \leq \kappa_i$ for all i. If in addition $C(x) = K$, then

$$\sum_i c_i P_i(x) = C(x) = K = \sum_i c_i\kappa_i,$$

and since $P_i(x) \leq \kappa_i$ termwise with positive weights, equality of the weighted sums forces $P_i(x) = \kappa_i$ for every i.

Now suppose $C(x) > K$. Using $K = \sum_i c_i\kappa_i$ and (J),

$$\sum_i c_i\Big(P_i(x) - \kappa_i\Big)^+ \;=\; C(x) - K \;=\; \sum_i c_i\Big(P_i(x) - \kappa_i\Big),$$

hence

$$\sum_i c_i\Big[\Big(P_i(x) - \kappa_i\Big)^+ - \Big(P_i(x) - \kappa_i\Big)\Big] = 0.$$

By the identity $a^+ - a = (-a)^+$, valid for all real a, this reads $\sum_i c_i\Big(\kappa_i - P_i(x)\Big)^+ = 0$, and the same positivity argument gives $P_i(x) \geq \kappa_i$ for all i. □

Remark 3.2 (Comonotonicity)**.** Geometrically, the lemma says that the curve $x \mapsto (P_1(x) - \kappa_1, \ldots, P_n(x) - \kappa_n)$ must take values in $\mathbb{R}^n_{\geq 0} \cup \mathbb{R}^n_{\leq 0}$, the union of the closed positive and negative orthants. This is a pathwise comonotonicity condition on the bond prices relative to the strike vector.

4. Exact Characterization

Theorem 4.1 (Exact obstruction to the Jamshidian decomposition)**.** *Let $d \geq 1$, let $E \subseteq \mathbb{R}^d$ be a connected open set, and let $P_i(x) = A_i e^{-B_i^\top x}$ with $A_i > 0$ and $B_i \in \mathbb{R}^d \setminus \{0\}$, $i = 1, \dots, n$. Fix cashflows $c_1, \dots, c_n > 0$ and a strike $K > 0$ satisfying* (NT)*. Then the following are equivalent.*

(1) *There exist deterministic strikes $\kappa_1, \dots, \kappa_n > 0$ with $\sum_i c_i \kappa_i = K$ such that* (J) *holds for every $x \in E$.*

(2) *The loading vectors are positively collinear: there exist $u \in \mathbb{R}^d \setminus \{0\}$ and scalars $b_1, \dots, b_n > 0$ with $B_i = b_i u$ for all i.*

Proof. **(2) ⇒ (1).** Without loss of generality $\|u\| = 1$. Consider the linear functional $x \mapsto u^\top x$ and set

$$I := \{\, u^\top x : x \in E \,\} \subseteq \mathbb{R}.$$

Since $u \neq 0$, the map $x \mapsto u^\top x$ is a surjective linear map onto $\mathbb{R}$ and hence open; I is therefore an open and, by connectedness of E, connected subset of $\mathbb{R}$, that is, an open interval.

Define $g(y) := \sum_{i=1}^n c_i A_i e^{-b_i y}$ for $y \in \mathbb{R}$. Each $b_i > 0$, so g is continuous and strictly decreasing, and $C(x) = g(u^\top x)$ for $x \in E$. Consequently $C(E) = g(I)$, and since g is a strictly decreasing homeomorphism onto its image, $g(I)$ is an open interval with

$$\inf g(I) = \inf_{x \in E} C(x), \qquad \sup g(I) = \sup_{x \in E} C(x).$$

Condition (NT) states precisely that $K \in g(I)$, so there is a unique $y^* \in I$ with $g(y^*) = K$. (Note that no assumption on the range of $u^\top$ over E beyond (NT) is needed: y^* is produced inside I, not on all of $\mathbb{R}$.)

Set $\kappa_i := A_i e^{-b_i y^*} > 0$; then $\sum_i c_i \kappa_i = g(y^*) = K$. For $x \in E$ write $y := u^\top x$. If $y \geq y^*$, then $P_i(x) = A_i e^{-b_i y} \leq \kappa_i$ for every i and $C(x) = g(y) \leq K$, so both sides of (J) vanish. If $y < y^*$, then $P_i(x) > \kappa_i$ for every i and $C(x) > K$, so

$$\sum_i c_i \Big(P_i(x) - \kappa_i\Big)^+ = \sum_i c_i \Big(P_i(x) - \kappa_i\Big) = C(x) - K = \Big(C(x) - K\Big)^+.$$

Thus (J) holds on all of E.

(1) ⇒ (2). Assume (J). Since E is connected and C is continuous, $C(E)$ is an interval; by (NT) it contains points on both sides of K, so there exists $x^* \in E$ with $C(x^*) = K$. Lemma 3.1 gives

$$P_i(x^*) = \kappa_i \qquad \text{for all } i. \tag{2}$$

Suppose, for contradiction, that some pair B_i, B_j is not positively collinear, and set

$$h := \frac{B_i}{\|B_i\|} - \frac{B_j}{\|B_j\|} \neq 0.$$

Failure of positive collinearity means $B_i^\top B_j < \|B_i\| \, \|B_j\|$: if B_i and B_j are not parallel this is the strict Cauchy–Schwarz inequality, and if they are anti-parallel the left side is negative. Hence

$$B_i^\top h = \|B_i\| - \frac{B_i^\top B_j}{\|B_j\|} > 0, \qquad B_j^\top h = \frac{B_i^\top B_j}{\|B_i\|} - \|B_j\| < 0.$$

Since E is open, $x_\varepsilon := x^* + \varepsilon h \in E$ for all sufficiently small $\varepsilon > 0$, and by (2)

$$P_i(x_\varepsilon) = \kappa_i \, e^{-\varepsilon B_i^\top h} < \kappa_i, \qquad P_j(x_\varepsilon) = \kappa_j \, e^{-\varepsilon B_j^\top h} > \kappa_j.$$

The shifted price vector at x_ε has components of both signs, contradicting Lemma 3.1.

Therefore every pair B_i, B_j is positively collinear. In particular each B_i is positively collinear with B_1: writing $u := B_1 / \|B_1\|$, there are scalars $b_i > 0$ with $B_i = b_i u$ for every i, which is (2). □

Corollary 4.2 (Impossibility in genuinely multifactor models)**.** *If* $\{B_i\}_{i=1}^n$ *spans a subspace of* $\mathbb{R}^d$ *of dimension at least* 2*, then no exact deterministic-strike Jamshidian decomposition exists for any strike satisfying* (NT)*.*

Remark 4.3 (Relation to the classical result)**.** For $d = 1$, condition (2) states that all the scalars B_i have the same sign, which holds in the standard one-factor short-rate models, where each $B_i > 0$ is a maturity-dependent duration coefficient. The sufficiency direction then reduces to Jamshidian's original construction [9]. The content of Theorem 4.1 is the converse: within the exponential-affine class nothing weaker than positive collinearity suffices, and the failure is detected by an explicit perturbation direction h along the exercise boundary.

5. Near-Collinearity and Projected Strikes

Theorem 4.1 is an all-or-nothing statement. In practice, loading vectors extracted from a calibrated multifactor model are often close to a common direction without lying on one, particularly over narrow maturity ranges. This section quantifies what survives of the decomposition in that regime. The results are of three kinds: a pathwise localization theorem showing that the deterministic-strike approximation errs only inside a strip around a projected exercise hyperplane (Theorem 5.3); a worst-case optimality result for the natural projected strikes (Proposition 5.4); and an upper bound on the pricing error under a regularity assumption on the factor distribution (Theorem 5.6). Throughout, the statements are upper bounds or worst-case statements; we do not claim, and do not use, any converse.

5.1. Transverse decomposition and dispersion

Fix a unit vector $u \in S^{d-1}$ such that

$$b_i := u^\top B_i > 0 \qquad (i = 1, \dots, n),$$

and write the orthogonal decompositions

$$B_i = b_i u + r_i, \quad r_i := B_i - b_i u \in u^\perp; \qquad x = yu + z, \quad y := u^\top x, \quad z := x - yu \in u^\perp.$$

Then $P_i(x) = A_i e^{-b_i y - r_i^\top z}$. Given deterministic strikes $\kappa_i > 0$ with $\sum_i c_i \kappa_i = K$, define

$$\tau_i := \frac{1}{b_i} \log \frac{A_i}{\kappa_i}, \qquad \alpha_i := \frac{r_i}{b_i}, \qquad \vartheta_i(z) := \tau_i - \alpha_i^\top z. \tag{3}$$

A direct computation gives $P_i(x) = \kappa_i\, e^{-b_i(y - \vartheta_i(z))}$, so

$$P_i(x) \geq \kappa_i \iff y \leq \vartheta_i(z) : \tag{4}$$

each component option has a *moving threshold* in the projected coordinate y, with slope $-\alpha_i$ in the transverse coordinate z.

Definition 5.1 (Envelopes, strip width, transverse dispersion)**.** For a strike vector κ define

$$L_\kappa(z) := \min_i \vartheta_i(z), \qquad U_\kappa(z) := \max_i \vartheta_i(z), \qquad W_\kappa(z) := U_\kappa(z) - L_\kappa(z) \geq 0,$$

and the *transverse dispersion* of the loadings relative to u,

$$D_u := \max_{i,j} \|\alpha_i - \alpha_j\| = \max_{i,j} \left\| \frac{r_i}{b_i} - \frac{r_j}{b_j} \right\|. \tag{5}$$

The *deterministic-strike Jamshidian approximation* associated with κ is

$$J_\kappa(x) := \sum_{i=1}^n c_i \Big(P_i(x) - \kappa_i \Big)^+.$$

Remark 5.2. $D_u = 0$ if and only if all α_i coincide, which holds if and only if the B_i are positively collinear (with common direction obtained by normalizing $u + \alpha_1$). Thus D_u vanishes exactly in the regime where Theorem 4.1 makes the decomposition exact, and D_u measures the angular spread of the loadings in the natural threshold coordinates.

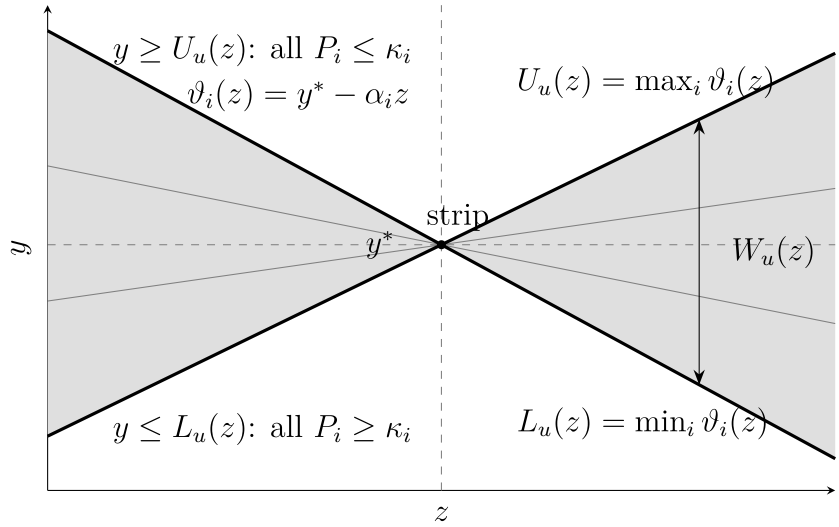

Figure 1: Strip localization with a one-dimensional transverse coordinate z, drawn for the projected strikes of Section 5.3, for which every moving threshold passes through the common point $(0, y^*)$. Outside the shaded strip all component options are exercised together and the approximation J_κ coincides with the true payoff; inside the strip the components disagree and J_κ exceeds the payoff by at most the amount in (6).

5.2. Strip localization and a pathwise bound

Theorem 5.3 (Strip localization and pathwise upper bound)**.** *Let $\kappa \in (0, \infty)^n$ be any deterministic strikes with $\sum_i c_i \kappa_i = K$. For every $x = yu + z \in E$,*

$$0 \;\leq\; J_\kappa(x) - \Big(C(x) - K\Big)^+ \;\leq\; \frac{K}{2}\Big(e^{b_{\max} W_\kappa(z)} - 1\Big)\,\mathbf{1}_{\{L_\kappa(z) \leq y \leq U_\kappa(z)\}}, \tag{6}$$

where $b_{\max} := \max_i b_i$. In particular the approximation is exact outside the strip:

$$J_\kappa(x) = \Big(C(x) - K\Big)^+ \qquad \textit{whenever } y \leq L_\kappa(z) \textit{ or } y \geq U_\kappa(z).$$

Proof. Set $a_i := c_i\Big(P_i(x) - \kappa_i\Big)$, so that $\sum_i a_i = C(x) - K$, $J_\kappa(x) = \sum_i a_i^+$, and $(C(x) - K)^+ = (\sum_i a_i)^+$. Subadditivity of $a \mapsto a^+$ gives the lower bound $J_\kappa(x) \geq (C(x) - K)^+$.

Exactness outside the strip. By (4), $a_i \geq 0$ if and only if $y \leq \vartheta_i(z)$. If $y \leq L_\kappa(z)$ then $a_i \geq 0$ for all i and $J_\kappa(x) = \sum_i a_i = C(x) - K = (C(x) - K)^+$. If $y \geq U_\kappa(z)$ then $a_i \leq 0$ for all i and both sides vanish.

Bound inside the strip. For real numbers $a_1, \ldots, a_n$, write $a_i = a_i^+ - a_i^-$. Then $\Big(\sum_i a_i\Big)^+ \geq \sum_i a_i^+ - \sum_i a_i^-$ and $\Big(\sum_i a_i\Big)^+ \geq 0$, whence

$$\sum_i a_i^+ - \Big(\sum_i a_i\Big)^+ \;\leq\; \min\Big\{\sum_i a_i^+,\, \sum_i a_i^-\Big\} \;\leq\; \frac{1}{2}\sum_i |a_i|\,.$$

If $L_\kappa(z) \le y \le U_\kappa(z)$, then $|y - \vartheta_i(z)| \le W_\kappa(z)$ for every i, so

$$|P_i(x) - \kappa_i| = \kappa_i\Big|e^{-b_i(y-\vartheta_i(z))} - 1\Big| \le \kappa_i\Big(e^{b_i W_\kappa(z)} - 1\Big) \le \kappa_i\Big(e^{b_{\max} W_\kappa(z)} - 1\Big).$$

Summing against the weights c_i and using $\sum_i c_i\kappa_i = K$ gives (6). □

5.3. Projected strikes and minimax optimality

The natural strikes to use in the multifactor setting are those produced by Jamshidian's construction applied to the projected scalar variable $y = u^\top x$. Since $y \mapsto \sum_i c_i A_i e^{-b_i y}$ is continuous, strictly decreasing, and maps $\mathbb{R}$ onto $(0,\infty)$, there is a unique $y^* \in \mathbb{R}$ with

$$\sum_{i=1}^{n} c_i A_i e^{-b_i y^*} = K, \tag{7}$$

and we define the *projected strikes*

$$\kappa_i^u := A_i e^{-b_i y^*}, \qquad i = 1, \dots, n. \tag{8}$$

Under the projected strikes $\tau_i \equiv y^*$, so the moving thresholds $\vartheta_i(z) = y^* - \alpha_i^\top z$ all pass through the common point $(0, y^*)$ and

$$W_u(z) := W_{\kappa^u}(z) = \max_i\Big(-\alpha_i^\top z\Big) - \min_i\Big(-\alpha_i^\top z\Big) \;\le\; D_u \,\|z\|\,. \tag{9}$$

We write $J_u := J_{\kappa^u}$, $L_u := L_{\kappa^u}$, and $U_u := U_{\kappa^u}$ for the corresponding approximation and envelopes.

Proposition 5.4 (Minimax optimality of the projected strikes)**.** *Fix u as above and $M > 0$. For every deterministic strike vector $\kappa \in (0,\infty)^n$ with $\sum_i c_i\kappa_i = K$,*

$$\sup_{\|z\|\le M} W_\kappa(z) \;\ge\; M D_u,$$

and the projected strikes κ^u attain equality. Moreover, κ^u is the unique minimizer whose thresholds $\tau_1, \dots, \tau_n$ in (3) *all coincide.*

Proof. From $\vartheta_i(z) = \tau_i - \alpha_i^\top z$,

$$W_\kappa(z) = \max_{i,j}\Big[(\tau_i - \tau_j) - (\alpha_i - \alpha_j)^\top z\Big].$$

For each ordered pair (i,j) the supremum of the bracket over $\|z\| \le M$ equals $(\tau_i - \tau_j) + M\,\|\alpha_i - \alpha_j\|$, and the supremum of a finite maximum is the maximum of the suprema, so

$$\sup_{\|z\|\le M} W_\kappa(z) = \max_{i,j}\Big[(\tau_i - \tau_j) + M\,\|\alpha_i - \alpha_j\|\Big]. \tag{10}$$

Let (i_0, j_0) attain D_u. Taking the larger of the (i_0, j_0) and (j_0, i_0) terms in (10) bounds the right side from below by $|\tau_{i_0} - \tau_{j_0}| + MD_u \geq MD_u$. For $\kappa = \kappa^u$ all τ_i equal y^*, and (10) evaluates to MD_u exactly.

Finally, suppose κ is a minimizer with a common threshold, $\tau_i \equiv \tau$. The budget constraint reads $\sum_i c_i A_i e^{-b_i \tau} = K$, whose unique solution is $\tau = y^*$ by strict monotonicity; hence $\kappa = \kappa^u$. □

Remark 5.5 (Non-uniqueness outside the common-threshold class)**.** The common-threshold restriction in the uniqueness statement is necessary. By (10), a strike vector attains the minimax value MD_u if and only if $|\tau_i - \tau_j| \leq M\Big(D_u - \|\alpha_i - \alpha_j\|\Big)$ for every pair. If some index k belongs to no pair attaining the maximum in (5), its threshold τ_k may be perturbed within these slack constraints while a common shift of the remaining thresholds restores the budget constraint; for $n \geq 3$ the minimizers then form a nontrivial family. The projected strikes are the canonical element of that family, and the only one whose component options share a single projected exercise threshold.

5.4. A pricing error bound

Prices are computed under the T_0-forward measure $\mathbb{Q}^{T_0}$:

$$V := P(0,T_0)\,\mathbb{E}^{T_0}\Big[\Big(C(X_{T_0}) - K\Big)^+\Big], \qquad V_u := P(0,T_0)\,\mathbb{E}^{T_0}\Big[J_u(X_{T_0})\Big],$$

where J_u is the projected-strike approximation of Section 5.3. Decompose $X_{T_0} = Yu + Z$ with $Y := u^\top X_{T_0}$ and $Z := X_{T_0} - Yu$.

Theorem 5.6 (Pricing error under a bounded conditional density)**.** *Assume that under $\mathbb{Q}^{T_0}$ the conditional law of Y given Z admits a density bounded by a constant L: $\Big\|f_{Y|Z}(\,\cdot \mid Z)\Big\|_\infty \leq L$ almost surely. Then*

$$0 \;\leq\; V_u - V \;\leq\; \frac{P(0,T_0)\,KL}{2}\,\mathbb{E}^{T_0}\Big[W_u(Z)\Big(e^{b_{\max} W_u(Z)} - 1\Big)\Big]. \tag{11}$$

If in addition $D_u \leq \bar{D}$ for some constant $\bar{D}$ and $\mathbb{E}^{T_0}\Big[\|Z\|^2\, e^{b_{\max}\bar{D}\|Z\|}\Big] < \infty$, then

$$0 \;\leq\; V_u - V \;\leq\; \frac{P(0,T_0)\,KL\,b_{\max}}{2}\,D_u^2\,\mathbb{E}^{T_0}\Big[\|Z\|^2\, e^{b_{\max}\bar{D}\|Z\|}\Big]. \tag{12}$$

In particular, $V_u - V = O(D_u^2)$ as $D_u \downarrow 0$ along any family of models for which L, $b_{\max}$, and the moment in (12) stay bounded.

Proof. By Theorem 5.3 applied with $\kappa = \kappa^u$,

$$0 \leq J_u(X_{T_0}) - \Big(C(X_{T_0}) - K\Big)^+ \leq \frac{K}{2}\Big(e^{b_{\max} W_u(Z)} - 1\Big)\mathbf{1}_{\{Y \in [L_u(Z),\,U_u(Z)]\}}.$$

Take $\mathbb{Q}^{T_0}$-expectations and condition on Z. The density bound gives $\mathbb{Q}^{T_0}\Big(Y \in [L_u(Z), U_u(Z)] \mid Z\Big) \leq L\,W_u(Z)$, which yields (11). For (12), use $W_u(Z) \leq D_u\,\|Z\|$

from (9) together with $e^a - 1 \le ae^a$ for $a \ge 0$:

$$W_u(Z)\Big(e^{b_{\max} W_u(Z)} - 1\Big) \le b_{\max} W_u(Z)^2 \, e^{b_{\max} W_u(Z)} \le b_{\max} D_u^2 \, \|Z\|^2 \, e^{b_{\max} \bar{D} \|Z\|}. \qquad \square$$

Remark 5.7 (On the assumptions)**.** Both assumptions are mild in the Gaussian affine class. If X_{T_0} is Gaussian under $\mathbb{Q}^{T_0}$, then (Y, Z) is jointly Gaussian, the conditional law of Y given Z is Gaussian with a variance $\sigma^2_{Y|Z}$ determined by the model parameters, and one may take $L = (2\pi\sigma^2_{Y|Z})^{-1/2}$ provided $\sigma^2_{Y|Z} > 0$; moreover $\|Z\|$ has Gaussian tails, so the exponential moment in (12) is finite for every $\bar{D}$. The moment condition can fail for heavy-tailed factor distributions; in that case (11) remains valid and only the $O(D_u^2)$ refinement is lost.

Corollary 5.8 (Angular form of the bound)**.** *Let $\theta_u := \max_i \angle(B_i, u)$ and $\delta := \max_{i,j} \angle(B_i, B_j)$, and suppose $\theta_u < \pi/2$. Then $D_u \le 2 \tan \theta_u$, and with the choice $u = B_1/\|B_1\|$ one has $\theta_u \le \delta$. Consequently, for any $\delta_0 < \pi/2$ and all configurations with $\delta \le \delta_0$,*

$$0 \;\le\; V_u - V \;\le\; 2\, P(0, T_0)\, K L\, b_{\max} \tan^2\!\delta \;\; \mathbb{E}^{T_0}\Big[\|Z\|^2 \, e^{2 b_{\max} \tan(\delta_0) \|Z\|}\Big], \tag{13}$$

so that $V_u - V = O(\delta^2)$ as $\delta \downarrow 0$ under the moment condition of Theorem 5.6.

Proof. Since $\|\alpha_i\| = \|r_i\| / b_i = \tan \angle(B_i, u)$, the triangle inequality gives $\|\alpha_i - \alpha_j\| \le \|\alpha_i\| + \|\alpha_j\| \le 2 \tan \theta_u$, hence $D_u \le 2 \tan \theta_u$. With $u = B_1/\|B_1\|$, $\angle(B_i, u) = \angle(B_i, B_1) \le \delta$ for every i, so $\theta_u \le \delta$. Substituting $\bar{D} = 2 \tan \delta_0$ and $D_u \le 2 \tan \delta$ into (12) gives (13). $\square$

Remark 5.9 (A two-factor scaling illustration)**.** The quadratic rate in (12) matches the natural perturbative regime. Take $B_1 = bu$ and $B_2^{\varepsilon} = bu + \varepsilon v$ with a unit vector $v \perp u$ and projected strikes along u. Then $D_u = \varepsilon / b$, the strip has width of order $(\varepsilon/b)\,|z|$, and the right side of (12) scales as ε^2. We present this as an illustration of consistency rather than a proof of sharpness; we have not established a matching lower bound for $V_u - V$, and the worked example below suggests that the constant in (11) is conservative in typical configurations.

6. A Worked Two-Factor Example

This section computes every object of Section 5 in a calibrated-style two-factor Gaussian model, both to illustrate the geometry and to report how the bound of Theorem 5.6 compares with the realized pricing error.

i	Δ_i	$B_i^{(1)}$	$B_i^{(2)}$	b_i	α_i	κ_i^u
1	1	0.6883	0.9610	1.1821	0.0000	0.9577
2	2	0.9976	1.8482	2.0835	0.1273	0.9201
3	3	1.1366	2.6672	2.8302	0.2223	0.8855
4	4	1.1990	3.4231	3.4811	0.2926	0.8530
5	5	1.2271	4.1210	4.0648	0.3449	0.8225

Table 1: Loadings, projected coefficients, transverse coordinates, and projected strikes in the two-factor example ($u = B_1/\|B_1\|$, $y^* = 0.011185$).

6.1. Specification

Take $d = 2$ and let the factors have Vasicek-type loadings with mean reversions $k_1 = 0.80$ and $k_2 = 0.08$:

$$B_i = \left(\tfrac{1-e^{-k_1\Delta_i}}{k_1},\ \tfrac{1-e^{-k_2\Delta_i}}{k_2}\right)^{\top}, \qquad \Delta_i := T_i - T_0,$$

the standard shape in two-factor Gaussian (G2-type) models. The option has expiry $T_0 = 1$ and annual payment tenors $\Delta_i = i$, $i = 1, \ldots, 5$, with cashflows $c_i = 0.04$ for $i \le 4$ and $c_5 = 1.04$ (a 4% annual coupon bond with unit principal). The constants $A_i = e^{-0.03\Delta_i}$ correspond to a flat 3% forward curve at expiry, and the strike is $K = 1$; the forward bond value at the central state is $C(0) = 1.0437$, so the option is modestly in the money there. Under $\mathbb{Q}^{T_0}$ the state is $X_{T_0} \sim \mathcal{N}(0, \Sigma)$ with

$$\Sigma_{11} = \sigma_1^2 \tfrac{1-e^{-2k_1T_0}}{2k_1}, \quad \Sigma_{22} = \sigma_2^2 \tfrac{1-e^{-2k_2T_0}}{2k_2}, \quad \Sigma_{12} = \rho\sigma_1\sigma_2 \tfrac{1-e^{-(k_1+k_2)T_0}}{k_1+k_2},$$

where $\sigma_1 = 1.2\%$, $\sigma_2 = 1.0\%$, $\rho = -0.7$, values typical of two-factor Gaussian calibrations. The model is a stylized specification of the expiry-date objects $(A_i, B_i, \mathcal{L}(X_{T_0}))$: no calibration to a particular market is intended, and Assumption 2.1 and the hypotheses of Theorem 5.6 hold exactly.

6.2. Geometry

Choose the reference direction $u = B_1/\|B_1\| = (0.5823,\ 0.8130)^{\top}$. Table 1 reports the loadings, the projections $b_i = u^{\top}B_i$, the scalar transverse coordinates α_i (signed, relative to the unit normal of u in the plane), and the projected strikes κ_i^u obtained from (7), which has solution $y^* = 0.011185$.

The transverse dispersion is $D_u = 0.3449$ and the maximal angle is $\theta_u = 0.3321$ radians, comfortably consistent with the bound $D_u \le 2\tan\theta_u = 0.6898$ of Corollary 5.8. The two mean-reversion speeds differ by a factor of ten, so this configuration is *not* fine-tuned to be nearly collinear; it is a representative two-factor geometry. Minimizing D_u over all admissible directions on the unit circle lowers it only from 0.3449 to 0.3352, so the convenient choice $u = B_1/\|B_1\|$ is within 3% of optimal here.

6.3. Prices, error, and the bound

In this Gaussian specification (Y, Z) is bivariate normal with $\mathrm{sd}(Y) = 0.00571$, $\mathrm{sd}(Z) = 0.01148$, and correlation 0.42; the conditional standard deviation of Y given Z is 0.00517, so Theorem 5.6 applies with $L = 77.20$.

Reference values were computed by conditioning on the transverse coordinate: given $Z = z$, all P_i are strictly decreasing in Y (each $b_i > 0$), so the conditional model is one-factor, Theorem 4.1 applies conditionally, and both $\mathbb{E}^{T_0}[(C-K)^+ \mid Z]$ and $\mathbb{E}^{T_0}[J_u \mid Z]$ reduce to sums of Black-type closed forms. The outer integral over z is one-dimensional and smooth and was evaluated by Gauss–Hermite quadrature with 150 nodes; the results were confirmed independently by tensorized two-dimensional quadrature and by Monte Carlo with 4×10^6 paths. In units of the T_0-forward numéraire (multiply by $P(0, T_0)$ for spot prices):

$$\mathbb{E}^{T_0}\Big[(C-K)^+\Big] = 0.0453411, \qquad \mathbb{E}^{T_0}[J_u] = 0.0453598,$$

so the projected approximation overprices, as Theorem 5.3 requires, by

$$\mathbb{E}^{T_0}[J_u] - \mathbb{E}^{T_0}\Big[(C-K)^+\Big] = 1.867 \times 10^{-5},$$

about 4 basis points of the option premium, even though $D_u = 0.34$ is far from the near-collinear regime. The probability that Y falls in the strip is 0.0652.

The right side of (11) evaluates to 2.49×10^{-3}, and the quadratic form (12) with $\bar{D} = D_u$ gives 2.52×10^{-3}: the bound holds with a factor of roughly 130 to spare. The conservatism has two identifiable sources, each visible in the proof of Theorem 5.6. First, the density bound estimates the strip probability by $L\,\mathbb{E}^{T_0}[W_u(Z)] = 0.244$ against an actual value of 0.065, since the bound ignores the rapid decay of the conditional density away from its mode. Second, inside the strip the amplitude estimate $\frac{K}{2}(e^{b_{\max} W_u(Z)} - 1)$ is a worst case attained only at the strip edges, whereas the actual gap $J_u - (C-K)^+$ vanishes at both edges and is small throughout most of the strip. The bound should accordingly be read as a structural statement about the scaling in D_u, not as a tight numerical estimate; in this example the realized error is itself negligible at the precision relevant for calibration.

Remark 6.1 (Diagnostic use)**.** The quantities in Table 1 require only the loading vectors, which are available in closed form in any affine Gaussian model. Computing D_u across the payment dates of a deal is therefore an immediate diagnostic for how far a calibrated multifactor model is from the exact Jamshidian regime, before any pricing is attempted.

7. Structural Extensions

Theorem 4.1 obstructs the decomposition for deterministic strikes. This section examines the three natural attempts to evade the obstruction: letting the strikes depend on the state, changing the numéraire, and replacing the linear projected factor by a nonlinear scalar factor. The first turns out to make the problem vacuous, and the other two leave the obstruction untouched. These results are boundary-of-definition clarifications rather than deep theorems; their role is to show that the deterministic-strike formulation of Definition 2.4 is exactly where the structural content of the problem lives.

7.1. State-dependent strikes make the problem vacuous

Proposition 7.1 (Proportional allocation)**.** *Let $P_i\colon E \to (0,\infty)$ be arbitrary positive functions, $c_i > 0$, $K > 0$, and $C(x) := \sum_i c_i P_i(x)$. Define*

$$\kappa_i(x) := \frac{K\,P_i(x)}{C(x)}, \qquad x \in E.$$

Then $\sum_i c_i \kappa_i(x) = K$ for every x, and

$$\Big(C(x) - K\Big)^+ = \sum_{i=1}^{n} c_i \Big(P_i(x) - \kappa_i(x)\Big)^+ \qquad \text{for all } x \in E. \tag{14}$$

Proof. The budget identity is immediate from the definition of C. For the payoff identity, note that

$$P_i(x) - \kappa_i(x) = P_i(x)\Big(1 - \frac{K}{C(x)}\Big),$$

so all n differences share the sign of $C(x) - K$. If $C(x) > K$ every positive part is the difference itself and the right side of (14) telescopes to $(1 - K/C(x))\,C(x) = C(x) - K$; if $C(x) \le K$ every positive part vanishes and both sides are zero. $\square$

Proposition 7.1 uses no structure whatsoever: no affinity, no monotonicity, no model. The question "does a state-dependent Jamshidian decomposition exist?" therefore has a trivially affirmative answer and no informational content. Any meaningful relaxation of Definition 2.4 must restrict the functional form of $\kappa_i(\cdot)$; we return to this in Section 8. Note also that the proportional allocation is invariant under deflation by any positive numéraire N, since $\kappa_i/N = K(P_i/N)/(C/N)$; restricting attention to numéraire-invariant adaptive strikes therefore does not restore content.

7.2. Numéraire changes do not alter the exercise geometry

Identity (J) is positively homogeneous in the pair (payoff, strikes), which makes it insensitive to deflation.

Lemma 7.2 (Deflation invariance)**.** *Let $N\colon E \to (0,\infty)$ be any strictly positive* *function. Then for all $x \in E$,*

$$\big(C(x)-K\big)^+ = N(x)\Big(\tfrac{C(x)}{N(x)} - \tfrac{K}{N(x)}\Big)^+, \qquad \{C>K\} = \Big\{\tfrac{C}{N} > \tfrac{K}{N}\Big\},$$

and (J) *holds for* (κ_i) *if and only if the deflated identity*

$$\frac{\big(C(x)-K\big)^+}{N(x)} = \sum_{i=1}^n c_i \, \frac{\big(P_i(x)-\kappa_i\big)^+}{N(x)}, \qquad x \in E,$$

holds for the same (κ_i).

Proof. All three statements follow from positive homogeneity of $t \mapsto t^+$ and strict positivity of N: $N(x)(t/N(x))^+ = t^+$ for every real t. ☐

Corollary 7.3 (The obstruction is geometric, not measure-theoretic)**.** *Suppose $\{B_i\}_{i=1}^n$ spans a subspace of dimension at least 2. Then for every strictly positive numéraire N there exist no deterministic strikes for which the N-deflated decomposition identity of Lemma 7.2 holds. In particular, no change of numéraire or of pricing measure can restore an exact decomposition of the original component assets with strikes that are deterministic in undeflated units: identity* (J) *is a pointwise property of the payoff functions on E, and deflation neither changes the exercise set $\{C>K\}$ nor the validity of the identity.*

Proof. Immediate from Lemma 7.2 and Corollary 4.2: the deflated identity for some (κ_i) is equivalent to (J) for the same (κ_i), which is impossible when the loadings span dimension at least 2. ☐

Remark 7.4. The change-of-numéraire technique of Geman, El Karoui and Rochet [7] remains, of course, central to the *valuation* of each component option once a decomposition (exact or approximate) is in hand. The corollary records only that it cannot create the decomposition: the obstruction lives in the factor-space geometry of $\{B_i\}$, not in the choice of pricing measure or units. A genuinely different relaxation, in which the component assets themselves are redefined as deflated ratios with strikes deterministic in the *deflated* units, changes the loading vectors to differences $B_i - B_0$ (for a bond numéraire with loading B_0) and falls outside the corollary: Theorem 4.1 applied to the deflated family shows that such a decomposition exists precisely when the differences $B_i - B_0$ are positively collinear. This relaxation is taken up in Section 8.

7.3. Scalar-factor rigidity

The sufficiency direction of Theorem 4.1 works because positive collinearity makes every bond price a monotone function of the *linear* scalar factor $u^\top x$. One might hope that in a non-collinear model some *nonlinear* scalar factor $g(x)$ could play the same role. The following result rules this out: for exponential-affine families, scalar representability through monotone links of any continuous scalar factor is already equivalent to positive collinearity.

For notational symmetry we state the result for $\Phi_i(x) = A_i e^{B_i^\top x}$; the bond-price form (1) is recovered by replacing B_i with $-B_i$, which preserves all three conditions below.

Theorem 7.5 (Scalar-factor rigidity). *Let $\Phi_i(x) = A_i e^{B_i^\top x}$ on a connected open set $E \subseteq \mathbb{R}^d$, with $A_i > 0$, $B_i \neq 0$, $i = 1, \ldots, n$. The following are equivalent.*

(i) *There exist a continuous map $g\colon E \to I$ onto an interval $I \subseteq \mathbb{R}$ and continuous functions $f_i\colon I \to (0, \infty)$, strictly monotone in the same direction, such that $\Phi_i = f_i \circ g$ for all i.*

(ii) *The family is pairwise pathwise comonotone:*

$$\Big(\Phi_i(x) - \Phi_i(y)\Big)\Big(\Phi_j(x) - \Phi_j(y)\Big) \geq 0 \qquad \text{for all } x, y \in E \text{ and all } i, j.$$

(iii) *The loadings are positively collinear: $B_i = b_i u$ for some $u \in \mathbb{R}^d \setminus \{0\}$ and $b_1, \ldots, b_n > 0$.*

Proof. We show (iii) ⇒ (i) ⇒ (ii) ⇒ (iii).

(iii) ⇒ (i)**.** Take $g(x) := u^\top x$. As in the proof of Theorem 4.1, $I := g(E)$ is an open interval and g is continuous onto I. With $f_i(t) := A_i e^{b_i t}$, strictly increasing since $b_i > 0$, we have $\Phi_i = f_i \circ g$ for every i.

(i) ⇒ (ii)**.** Fix $x, y \in E$ and i, j. If $g(x) \geq g(y)$ then, since all f_k are monotone in the same direction, the differences $\Phi_i(x) - \Phi_i(y)$ and $\Phi_j(x) - \Phi_j(y)$ have the same sign (both ≥ 0 if the f_k are increasing, both ≤ 0 otherwise); symmetrically for $g(x) \leq g(y)$. In all cases the product is nonnegative. Continuity of g and the f_i is not used in this step; monotone f_i and a common g suffice.

(ii) ⇒ (iii)**.** Suppose some pair B_i, B_j is not positively collinear and set $v := B_i/\|B_i\| - B_j/\|B_j\| \neq 0$. Exactly as in the proof of Theorem 4.1, $B_i^\top v > 0 > B_j^\top v$. Fix any $x^* \in E$; since E is open, $x := x^* + \varepsilon v \in E$ for small $\varepsilon > 0$. With $y := x^*$,

$$\Phi_i(x) - \Phi_i(y) = \Phi_i(x^*)\Big(e^{\varepsilon B_i^\top v} - 1\Big) > 0, \qquad \Phi_j(x) - \Phi_j(y) = \Phi_j(x^*)\Big(e^{\varepsilon B_j^\top v} - 1\Big) < 0,$$

so the product is negative, contradicting (ii). Hence every pair is positively collinear, and as in the proof of Theorem 4.1, $u := B_1/\|B_1\|$ and suitable $b_i > 0$ give $B_i = b_i u$ for all i. □

Corollary 7.6 (Nonlinear scalar factors have no extra power). *For an exponential-affine family, the existence of* any *scalar-factor representation $\Phi_i = f_i \circ g$ with common continuous g and co-monotone links imposes exactly the same condition on the loadings as the linear choice $g(x) = u^\top x$, namely positive collinearity. In* *particular, when Theorem 4.1 obstructs the decomposition, no change of state variable, however nonlinear, removes the obstruction.*

Remark 7.7 (Measurability)**.** The continuity assumptions enter only in the implication (iii) $\Rightarrow$ (i), where the constructed g is in fact linear. As noted in the proof, (i) $\Rightarrow$ (ii) requires only that the f_i be monotone in the same direction and share a common g, with no regularity on g; and (ii) $\Rightarrow$ (iii) makes no reference to g at all. Hence allowing merely Borel-measurable scalar factors enlarges condition (i) but not the set of loading configurations satisfying it: comonotonicity still forces positive collinearity.

8. Discussion and Open Problems

8.1. Geometric reading of the obstruction

The map $x \mapsto (P_1(x), \dots, P_n(x))$ traces a d-parameter surface $\mathcal{S}$ in the positive orthant of $\mathbb{R}^n$. Identity (J) asks the affine hyperplane $\{\sum_i c_i z_i = K\}$ to split $\mathcal{S}$ into the two pieces $\mathcal{S} \cap \{z \geq \kappa\}$ and $\mathcal{S} \cap \{z \leq \kappa\}$, where the inequalities are componentwise and κ is the strike vector. When the loadings are positively collinear, $\mathcal{S}$ is a curve that crosses the hyperplane at the single point κ, and the split is achievable. When the loadings span two or more dimensions, the perturbation direction h in the proof of Theorem 4.1 moves along $\mathcal{S}$ near the crossing point in a way that raises some coordinates and lowers others, and no strike vector on the hyperplane can separate the surface. The failure is thus a transversality phenomenon, which is why no reparameterization of the domain (Theorem 7.5) and no deflation of the range (Corollary 7.3) can repair it.

8.2. Open problems

Four questions seem natural. First, Proposition 5.4 establishes optimality of the projected strikes for worst-case strip width; whether they also minimize the *expected* pricing error $\mathbb{E}^{T_0}[J_\kappa] - V$ for a fixed factor distribution, and if not by how much they fail to, is open. Second, the bounds depend on the reference direction u through D_u, L, and the law of Z; minimizing the bound over $u \in S^{d-1}$ is a finite-dimensional problem whose solution we have not characterized, though the example of Section 6 suggests that simple choices are nearly optimal in well-conditioned configurations. Third, Proposition 7.1 shows that unrestricted state-dependent strikes trivialize the problem, while Theorem 4.1 shows that constant strikes are obstructed; it would be interesting to identify restricted adaptive classes, for instance strikes measurable with respect to a single linear functional of the state, or decompositions in deflated coordinates where the loadings become differences $B_i - B_0$, for which existence is nontrivial in both directions. Fourth, the exponential-affine form (1) is essential to our proofs; extending the characterization to quadratic Gaussian models, where $\log P_i$ is quadratic in the state, would require controlling a genuinely nonlinear exercise boundary and appears to demand different tools.

8.3. Concluding remarks

Within the exponential-affine class, the question "when does Jamshidian's trick work?" has an exact answer: precisely when the loading vectors lie on a common positive ray (Theorem 4.1). The answer is stable under the natural attempts to evade it, and it degrades gracefully: near the collinear regime, the projected one-factor strikes give a pathwise upper bound with error localized in an explicit strip (Theorem 5.3), worst-case optimal width (Proposition 5.4), and pricing error controlled by the square of a computable dispersion coefficient (Theorem 5.6). The worked example indicates that the dispersion diagnostic is easy to evaluate in calibrated models and that the realized projection error can be small even well outside the asymptotic regime.

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Replication. A short script reproducing all quantities reported in Section 6 is available from the author.

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